ΜΕΜ-205 Περιγραφική Στατιστική

Τμήμα Μαθηματικών και Εφ. Μαθηματικών, Πανεπιστήμιο Κρήτης

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11-05-2020

Συντελεστές Αυτοσυσχέτισης - Παράδειγμα

$Y_1 \quad Y_2$

$$\{5,\ 3,\ 2,\ 3,\ 5,\ 4,\ 2,\ 2\}, \qquad \text{PACF}(2) = \dots \qquad \text{PACF}(3) = \dots$$

#1 $\quad Y_{n-1} \to Y_n \quad Y_n = A_1 + B_1 Y_{n-1} + \varepsilon_{n-1}^{(1)} \quad n = 2, \dots, 8$

$\{(5,3), (3,2), (2,3), (3,5), (5,4), (4,2), (2,2)\}$

#2 $\quad Y_{n-1} \to Y_{n-2} \quad Y_{n-2} = A_2 + B_2 Y_{n-1} + \varepsilon_{n-1}^{(2)} \quad n = 3, \dots, 9$

$\{(3,5), (2,3), (3,2), (5,3), (4,5), (2,4), (2,2)\}$

$$b = \frac{SS_{xy}}{SS_{xx}} \qquad SS_{xy} = \sum_n (x_n - \bar{X})(y_n - \bar{Y}) \qquad SS_{xx} = \sum_n (x_n - \bar{X})^2 \qquad a = \bar{Y} - b\bar{X}$$

#1 $\quad b_1 = \dfrac{SS_{Y_{n-1}, Y_n}}{SS_{Y_{n-1}, Y_{n-1}}} \qquad a = \bar{Y}_n - b\bar{Y}_{n-1} \qquad \bar{Y}_{n-1} = 3.43 \qquad \bar{Y}_n = 3$

$SS_{Y_{n-1}, Y_{n-1}} = (5-3.43)^2 + (3-3.43)^2 + \dots + (2-3.43)^2 = 9.71$

$SS_{Y_{n-1}, Y_n} = (5-3.43)(3-3) + (3-3.43)(2-3) + \dots + (2-3.43)(2-3)$
$= -2.3349$

$b_1 = \dfrac{-2.3349}{9.71} = -0.24 \qquad a = 3 + 0.24 \cdot 3.43 = 3.82$

$$\boxed{\hat{y}_n = 3.82 - 0.24\, y_{n-1}}$$

$\hat{y}_2 = 3.82 - 0.24\, y_1 = 3.82 - 0.24 \cdot 5 = 2.62$

#2 $\quad b_1 = \dfrac{SS_{Y_{n-1}, Y_{n-2}}}{SS_{Y_{n-1}, n-1}} \qquad a = \bar{Y}_{n-2} - b\bar{Y}_{n-1} \qquad \bar{Y}_{n-1} = 3 \qquad \bar{Y}_{n-2} = 3.43$

$b_1 = -0.29 \qquad a = 4.3$

$$\boxed{\hat{y}_{n-2} = 4.3 - 0.29\, y_{n-1}}$$

Συντελεστές Αυτοσυσχέτισης - Παράδειγμα

$$\{5,\ 3,\ 2,\ 3,\ 5,\ 4,\ 2,\ 2\}, \qquad \text{PACF}(2) = \dots \quad \text{PACF}(3) = \dots$$

Kalman Gain - Παράδειγμα

$$x_n = -x_{n-1} + \epsilon_{n-1},\ \sigma_\epsilon^2 = 20, \quad \sigma^2_{\hat{x}_{0,0}} = 30$$

$$z_n \sim \mathcal{N}(x_n, \sigma_z^2),\ \sigma_z^2 = 10$$

	y_n	$\hat{y}_n$	$e_n^{(1)} = y_n - \hat{y}_n$	
y_2	3	2.62	0.38	$e_2^{(1)}$
y_3	2	3.1	-1.1	$e_3^{(1)}$
y_4	3	3.34	-0.34	
y_5	5	3.1	1.9	
y_6	4	2.62	1.38	
y_7	2	2.86	-0.86	
y_8	2	3.34	-1.34	$e_8^{(1)}$

	y_{n-2}	$\hat{y}_{n-2}$	$e_{n-2}^{(2)}$	
y_1	5	3.43	1.57	$= e_1^{(2)}$
y_2	3	3.72	-0.72	$= e_2^{(2)}$
y_3	2	3.42	-1.42	
y_4	3	2.85	0.15	
y_5	5	3.14	1.86	
y_6	4	3.72	0.28	
y_7	2	3.72	-1.72	$= e_7^{(2)}$

$e_n^{(1)} = A_3 + B_3 e_{n-2}^{(2)} + \varepsilon^{(3)}$ $\quad n=3$ το πρώτο στοιχείο

$\{(\overset{x}{1.57}, \overset{y}{-1.1}), (-0.72, -0.34), \dots (0.28, -1.34)\}$ $\quad 6$

$$PACF(2) = \frac{SS_{e^{(1)}e^{(2)}}}{\sqrt{SS_{e^{(1)}e^{(1)}} SS_{e^{(2)}e^{(2)}}}} =$$

$$= -0.76$$

$SS_{e^{(1)}e^{(2)}} = 1.57 \cdot (-1.1) + (-0.72) \cdot (-0.34) + \dots + 0.28 \cdot (-1.34)$

$= -7.808$

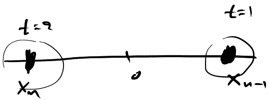
$SS_{e^{(2)}e^{(2)}} = 1.57^2 + (-0.72)^2 + \dots + 0.28^2 = 8.56$

$SS_{e^{(1)}e^{(1)}} = (-1.1)^2 + (-0.34)^2 + \dots + (-1.34) = 12.095$

Kalman Gain - Παράδειγμα

$$x_n = -x_{n-1} + \epsilon_{n-1},\ \sigma^2_{\epsilon} = 20, \quad \sigma^2_{\hat{x}_{0,0}} = 30 \qquad \varepsilon_n \sim \mathcal{N}(0,20)$$

$$z_n \sim \mathcal{N}(x_n, \sigma_z^2),\ \sigma_z^2 = 10$$



$k_n = ;$ $\quad n=1,2,3$

$\sigma^2_{\hat{x}_{n,n}}$

$$\hat{x}_{n-1,n-1} \longrightarrow \hat{x}_{n,n-1} \qquad \hat{x}_{n,n-1} = -\hat{x}_{n-1,n-1}$$

$$\boxed{\sigma^2_{\hat{x}_{n,n-1}} = \sigma^2_{\hat{x}_{n-1,n-1}} + \sigma^2_{\varepsilon}} \quad (1)$$

t_{n-1}: $\quad \hat{x}_{n,n} = \hat{x}_{n,n-1} + k_n (z_n - \hat{x}_{n,n-1})$

$$\boxed{k_n = \frac{\sigma^2_{\hat{x}_{n,n-1}}}{\sigma^2_{\hat{x}_{n,n-1}} + \sigma^2_z}} \quad (2)$$

t_n:

$$\boxed{\sigma^2_{\hat{x}_{n,n}} = (1 - k_n)^2 \sigma^2_{\hat{x}_{n,n-1}}} \quad (3)$$

$$k_1 = \frac{\sigma^2_{\hat{x}_{1,0}}}{\sigma^2_{\hat{x}_{1,0}} + \sigma^2_z} = \frac{50}{50+10} = \frac{50}{60} = 5/6$$

$(1)\ n=1 \Rightarrow \sigma^2_{\hat{x}_{1,0}} = \sigma^2_{\hat{x}_{0,0}} + \sigma^2_{\varepsilon} = 30 + 20 = 50$

$(3) + \sigma^2_{\hat{x}_{1,0}} = 50 \Rightarrow \sigma^2_{\hat{x}_{1,1}} = (1 - 5/6)^2 \cdot 50 = \frac{1}{36} \cdot 50 = 1.38$

Kalman Gain - Παράδειγμα

$$x_n = -x_{n-1} + \epsilon_{n-1},\ \sigma_\epsilon^2 = 20, \quad \sigma_{x_{0,0}}^2 = 30$$

$$z_n \sim \mathcal{N}(x_n, \sigma_z^2),\ \sigma_z^2 = 10$$

$$\{1, 2, 3, 4, 5, 6\}$$

$$f(t) = \frac{\beta_3}{1 + \beta_2 \exp\{-\beta_1 t\}}$$

$$\left\{ \left(\frac{1}{y_1}, \frac{1}{y_2}\right), \left(\frac{1}{y_2}, \frac{1}{y_3}\right), \ldots, \left(\frac{1}{y_5}, \frac{1}{y_6}\right) \right\}$$

$$\left\{ \left(\overset{x}{1}, \overset{y}{\frac{1}{2}}\right), \left(\frac{1}{2}, \frac{1}{3}\right), \ldots, \left(\frac{1}{5}, \frac{1}{6}\right) \right\}$$

$$\hat{y} = \alpha + bx \qquad \bar{x} \qquad \bar{y}$$

$$b = \ldots$$

$$\alpha = \ldots$$

$$\hat{\beta}_1 = -\ln b$$

$$\hat{\beta}_3 = \frac{1 - e^{-\hat{\beta}_1}}{\hat{a}}$$

$$\beta_2 = \left(\frac{\hat{\beta}_3}{1} - 1\right) \exp\{\hat{\beta}_1\}$$

Kalman Gain - Παράδειγμα

$$x_n = -x_{n-1} + \epsilon_{n-1},\ \sigma_\epsilon^2 = 20, \quad \sigma_{\hat{x}_{0,0}}^2 = 30$$
$$z_n \sim \mathcal{N}(x_n, \sigma_z^2),\ \sigma_z^2 = 10$$

Kalman Gain - Παράδειγμα

$$x_n = -x_{n-1} + \epsilon_{n-1},\ \sigma_\epsilon^2 = 20, \quad \sigma_{\hat{x}_{0,0}}^2 = 30$$

$$z_n \sim \mathcal{N}(x_n, \sigma_z^2),\ \sigma_z^2 = 10$$

$\{5, 3, 2, 3, 5, 4, 2, 2\}$ PACF(3)

$$\begin{pmatrix} Y_{n-k+1}, \dots, Y_{n-1} \to Y_n \\ Y_{n-k+1}, \dots, Y_{n-1} \to Y_{n-k} \end{pmatrix}$$

#1 $Y_{n-2}, Y_{n-1} \to Y_n$ $\{(5,3,2), (3,2,3), \dots, (4,2,2)\}$ 6

#2 $Y_{n-2}, Y_{n-1} \to Y_{n-3}$ $\{(3,2,5), (2,3,3), (3,5,2), (5,4,3), (4,2,5), (2,2,4)\}$

#1

$$X = \begin{bmatrix} 1 & 5 & 3 \\ 1 & 3 & 2 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 3 & 4 \\ 1 & 4 & 2 \end{bmatrix} \quad y = \begin{bmatrix} 2 \\ 3 \\ 5 \\ 4 \\ 2 \\ 2 \end{bmatrix} \quad \beta = (X^T X)^{-1} X^T y = \begin{bmatrix} 5.48 \\ -0.97 \\ 0.34 \end{bmatrix}$$

$$\hat{y}_n = 5.48 - 0.47\, y_{n-2} + 0.34\, y_{n-1}$$

#2

$$X = \begin{bmatrix} 1 & 3 & 2 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 4 \\ 1 & 4 & 2 \\ 1 & 2 & 2 \end{bmatrix} \quad y = \begin{bmatrix} 5 \\ 3 \\ 2 \\ 3 \\ 5 \\ 4 \end{bmatrix}$$

$$\hat{y}_{n-3} = 5.5 + 0.33\, y_{n-2} - 0.96\, y_{n-1}$$

	y	$\hat{y}$	$e^{(1)}$
y_3	2	1.65	0.35
y_4	3	3.25	-0.25
y_5	5	4.56	0.44
y_6	4	4.27	-0.27
y_7	2	1.99	0.01
y_8	2	2.28	-0.28

	y	$\hat{y}$	$e^{(2)}$
y_1	5	4.57	0.43
y_2	3	3.28	-0.28
y_3	2	1.69	0.31
y_4	3	3.31	-0.31
y_5	5	4.9	0.1
y_6	4	4.24	-0.24

$$e^{(2)}_{n-3} \to e^{(1)}_n$$

$$PACF(3) = \frac{SS_{e^{(1)}e^{(2)}}}{\sqrt{SS_{e^{(1)}e^{(1)}}\, SS_{e^{(2)}e^{(2)}}}} = -0.65$$

$$SS_{e^{(1)}e^{(1)}} = (-0.25)^2 + 0.44^2 + \dots$$

$$SS_{e^{(2)}e^{(2)}} = 0.43^2 + (-0.28)^2 + \dots$$

$\{(0.43, -0.25), (-0.28, 0.44), \dots, (0.1, -0.28)\}$

$$SS_{e^{(1)}e^{(2)}} = 0.43 \cdot (-0.25) + (-0.28) \cdot 0.44 + \dots$$