MEM-284: Κυματική Διάδοση

Τμήμα Μαθηματικών και Εφαρμοσμένων Μαθηματικών, Πανεπιστήμιο Κρήτης

Κώστας Σμαραγδάκης (https://kesmarag.gitlab.io)

1η Διάλεξη - 17.2.2022

Πληροφορίες για το μάθημα

Ιστοσελίδα του μαθήματος

https://kesmarag.gitlab.io/courses/mem284/

Ώρες γραφείου

Θα ανακοινωθούν την επόμενη εβδομάδα.

Αξιολόγηση

- 2 εργαστηριακές ασκήσεις - 30 %
  - θεωρητικές και υπολογιστικές ασκήσεις
  - Υποχρεωτικές για την συμμετοχή στη τελική εξέταση (Ιουνίου και Σεπτεμβρίου).
  - Προφορική και γραπτή εξέταση.
- Τελική εξέταση - 70 %
  - Θέματα ανάπτυξης με κλειστές σημειώσεις.
  - Πρέπει ο βαθμός της τελικής εξέτασης να είναι τουλάχιστον 4.

Προαπαιτούμενες γνώσεις

Για να παρακολουθήσετε και να επιτύχετε στο μάθημα είναι απαραίτητο να έχετε αρκετά καλό επίπεδο στα επόμενα μαθήματα.

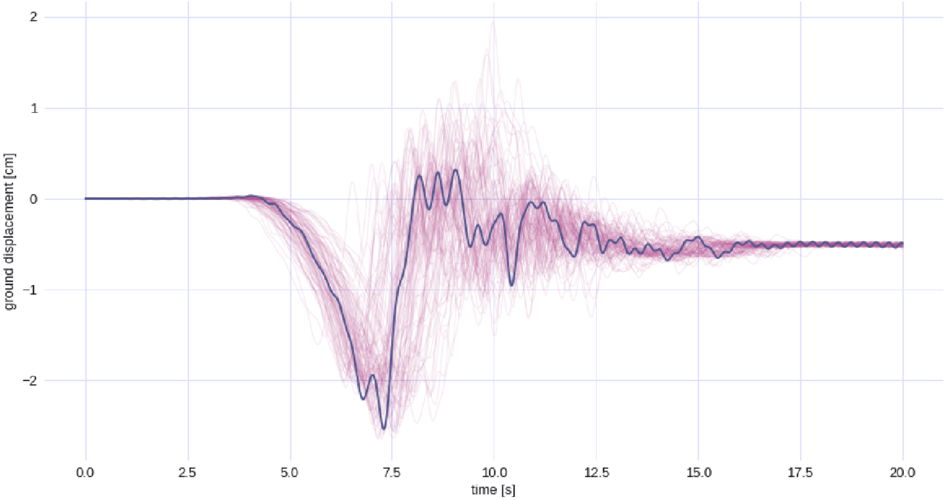
- Απειροστικός λογισμός I,II,III.
- Εισαγωγή στη γραμμική άλγεβρα.
- Γλώσσα προγραμματισμού I.

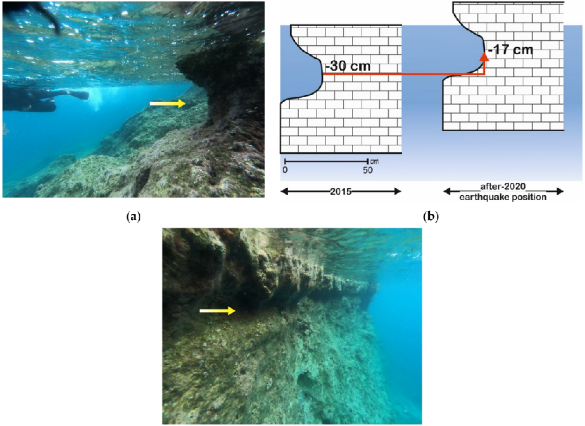
Πληροφορίες για το μάθημα

Το μάθημα αποτελεί μια εισαγωγή στη κυματική διάδοση. Εμείς θα ασχοληθούμε με το πρόβλημα της κυματικής διάδοσης (1D, 2D, 3D) σε ελαστικά μέσα.

- Άμεση εφαρμογή: Σεισμολογία

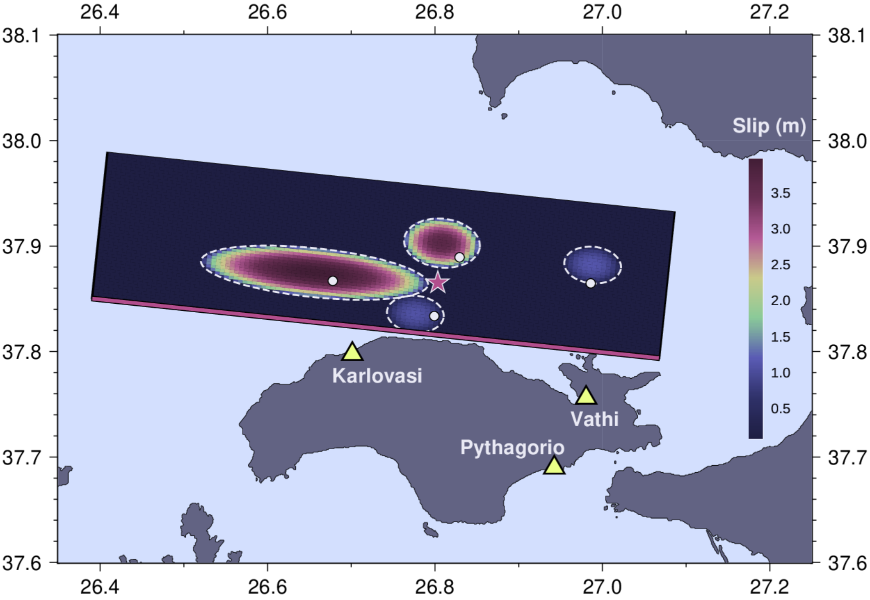
Μελέτη δονήσεων στο εσωτερικό και στη επιφάνεια της Γής.

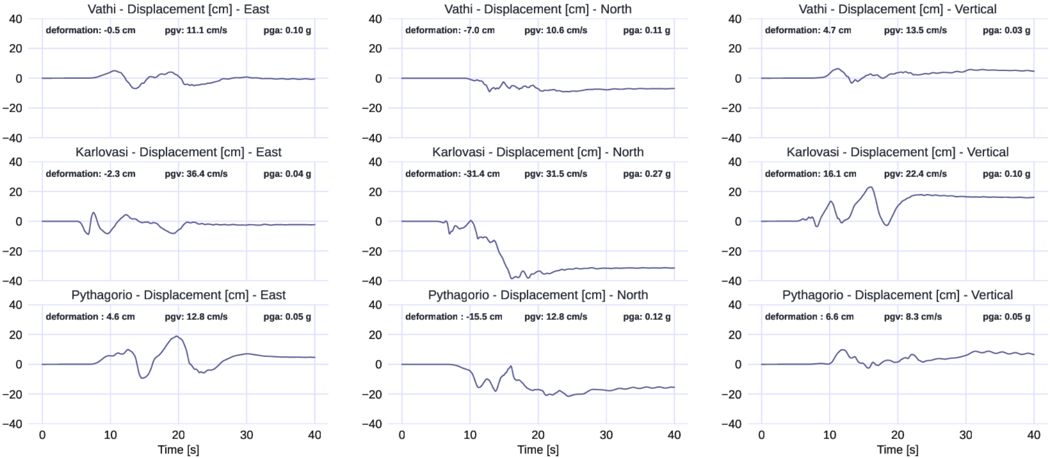

2
1
0
−1
−2
ground displacement [cm]
0.0
2.5
5.0
7.5
10.0
12.5
15.0
17.5
20.0
time [s]

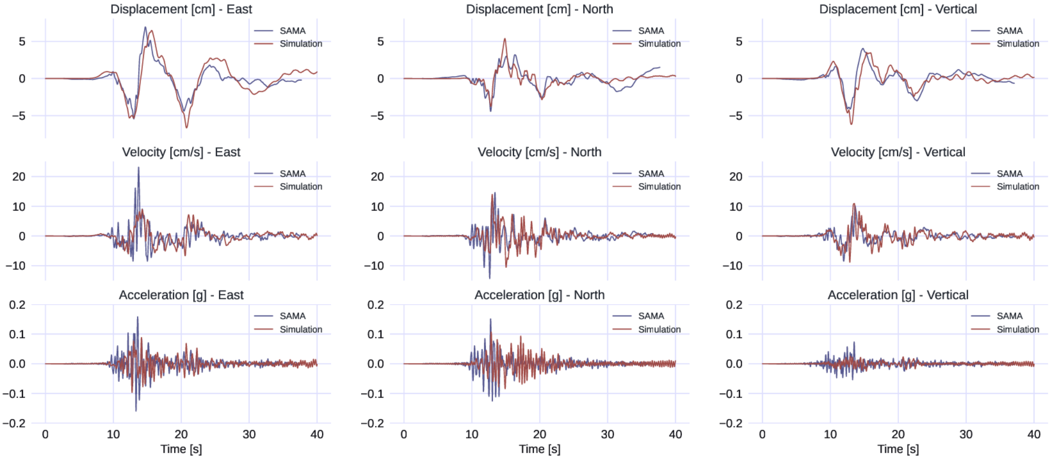


(a) (b) (c)

- Evelpidou et al, J. Mar. Sci. Eng. 2021


26.4
26.6
26.8
27.0
27.2
38.1
38.0
37.9
37.8
37.7
37.6
Slip (m)
3.5
3.0
2.5
2.0
1.5
1.0
0.5
Karlovasi
Vathi
Pythagorio


Vathi - Displacement [cm] - East
deformation: -0.5 cm
pgv: 11.1 cm/s
pga: 0.10 g
Vathi - Displacement [cm] - North
deformation: -7.0 cm
pgv: 10.6 cm/s
pga: 0.11 g
Vathi - Displacement [cm] - Vertical
deformation: 4.7 cm
pgv: 13.5 cm/s
pga: 0.03 g
Karlovasi - Displacement [cm] - East
deformation: -2.3 cm
pgv: 36.4 cm/s
pga: 0.04 g
Karlovasi - Displacement [cm] - North
deformation: -31.4 cm
pgv: 31.5 cm/s
pga: 0.27 g
Karlovasi - Displacement [cm] - Vertical
deformation: 16.1 cm
pgv: 22.4 cm/s
pga: 0.10 g
Pythagorio - Displacement [cm] - East
deformation : 4.6 cm
pgv: 12.8 cm/s
pga: 0.05 g
Pythagorio - Displacement [cm] - North
deformation : -15.5 cm
pgv: 12.8 cm/s
pga: 0.12 g
Pythagorio - Displacement [cm] - Vertical
deformation : 6.6 cm
pgv: 8.3 cm/s
pga: 0.05 g
40
20
0
−20
−40
0
10
20
30
40
Time [s]


Displacement [cm] - East
Displacement [cm] - North
Displacement [cm] - Vertical
Velocity [cm/s] - East
Velocity [cm/s] - North
Velocity [cm/s] - Vertical
Acceleration [g] - East
Acceleration [g] - North
Acceleration [g] - Vertical
SAMA
Simulation
Time [s]

$u u_x$ $\qquad u_{xx} - c^{-2} u_{tt} = 0 \qquad u'' - c^{-2}\ddot{u} = 0$

$$\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u(x,t)}{\partial t^2} = 0$$

Αρχή της υπέρθεσης

- Έστω u_1, u_2 λύσεις της μονοδιάστατης κυματικής εξίσωσης τότε $au_1 + bu_2$ αποτελεί επίσης λύση.
- Έστω το σύνολο των λύσεων $\{u(x,t;\lambda),\ \lambda \in \mathcal{I} \subset \mathbb{R}\}$ τότε

$$\int_{\mathcal{I}} u(x,t;\lambda)d\lambda$$

αποτελεί επίσης λύση.

$$\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u(x,t)}{\partial t^2} = 0$$

Θα δείξουμε ότι για τυχαίες συναρτήσεις f, g, τα u_1, u_2 αποτελούν λύσεις της εξίσωσης.

$$u_1(x,t) = f(x-ct), \quad u_2(x,t) = g(x+ct), \;\; c > 0$$

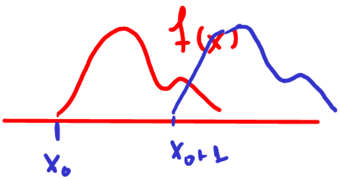
$x \pm ct$ χαρακτηριστικές τιμές.

☻ f, g όχι απαραίτητα παραγωγίσιμες ☻

Σημείωση

$\frac{\partial}{\partial x} f = \partial_x f$

- $f(x-ct)$ διαδίδεται προς τα δεξιά με σταθερή ταχύτητα c.
- $g(x+ct)$ διαδίδεται προς τα αριστερά με σταθερή ταχύτητα c.



$f(x-1)$

$\partial_x u_1 = f'(\overbrace{x-ct}^{\xi})$ $\quad$ $\partial_{xx} u_1 = f''(x-ct)$

$\partial_t u_1 = -c f'(x-ct)$ $\quad$ $\partial_{tt} u_1 = c^2 f''(x-ct)$

- Αποτελεί μια γενικευμένη συνάρτηση η οποία ορίζεται μέσω της σχέσης

$$\int_{-\infty}^{\infty} \delta(x - x_0) f(x) dx = f(x_0)$$

οπου f μπορεί πρακτικά να είναι μια τυχούσα συνάρτηση.

Έστω $f(x) = 1$

$$\int_{-\infty}^{\infty} \delta(x - x_0) dx = 1$$

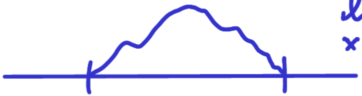
Οδεύων παλμός Dirac

$$u(x,t) = \delta(x - ct)$$

$u_{xx} - c^{-2} u_{tt} = 0$

test function $\varphi \in C_0^\infty$

$(u_{xx} - c^{-2} u_{tt})\varphi(x,t) = 0$



$\lim_{x \to \pm\infty} \varphi = 0$

$\iint (u_{xx} - c^{-2} u_{tt})\varphi(x,t)\,dx\,dt = 0 \quad \forall \varphi \in C_0^\infty$

$\iint u_{xx}\varphi(x,t)\,dx\,dt = c^{-2}\iint u_{tt}\varphi(x,t)\,dx\,dt$

$\int_a^b f'g\,dx = \overbrace{[fg]_a^b}^{0} - \int fg'\,dx$

$\iint u_{xx}\varphi\,dx\,dt = -\iint u_x\varphi_x\,dx\,dt = \iint u\varphi_{xx}\,dx\,dt$

$$c^{-2}\iint u_{tt}\,\varphi\,dx\,dt = c^{-2}\iint u\,\varphi_{tt}\,dx\,dt$$

$$\iint u\,\varphi_{xx}\,dx\,dt = c^{-2}\iint u\,\varphi_{tt}\,dx\,dt \quad \forall \varphi \in C_0^{\infty}$$

$$\iint \delta(x-ct)\,\overbrace{\varphi_{xx}(x,t)}^{f(x,t)}\,dx\,dt = \int_{-\infty}^{+\infty} \varphi_{xx}(ct,t)\,dt$$

Θ.δ.ο =

$$c^{-2}\iint \delta(x-ct)\,\varphi_{tt}(x,t)\,dx\,dt = c^{-2}\int_{-\infty}^{+\infty} \varphi_{tt}(ct,t)\,dt$$

$\xi = ct$

$$\int_{-\infty}^{+\infty} \varphi_{xx}(ct,t)\,dt = \lim_{h\to 0} \int_{-\infty}^{+\infty} dt \, \frac{\varphi(\overbrace{ct-h}^{\xi},t) - 2\varphi(ct,t) + \varphi(ct+h,t)}{h^2}$$

$$\xi = ct \pm h \quad , \quad d\xi = c\,dt$$

$$= \frac{1}{c} \lim_{h\to 0} \int_{-\infty}^{+\infty} d\xi \, \frac{\varphi\left(\xi, \frac{\xi+h}{c}\right) - 2\varphi\left(\xi, \frac{\xi}{c}\right) + \varphi\left(\xi, \frac{\xi-h}{c}\right)}{h^2} =$$

$$= \frac{1}{c} \lim_{\frac{h}{c}\to 0} \int d\xi \, \frac{\varphi\left(\xi, \frac{\xi}{c}+\frac{h}{c}\right) - 2\varphi\left(\xi, \frac{\xi}{c}\right) + \varphi\left(\xi, \frac{\xi}{c}-\frac{h}{c}\right)}{c^2 \frac{h^2}{c^2}} =$$

$$= \frac{1}{c^3} \lim_{h^* \to 0} \int d\xi \, \frac{\varphi(\xi, \xi/c + h^*) - 2\varphi(\xi, \xi/c) + \varphi(\xi, \xi/c - h^*)}{(h^*)^2} =$$

$$= \frac{1}{c^3} \int_{-\infty}^{+\infty} \varphi_{tt}(\xi, \overset{t}{\xi/c}) \, d\xi = \frac{\cancel{c}}{c^{\cancel{3}2}} \int_{-\infty}^{+\infty} \varphi_{tt}(ct, t) \, dt$$

$$d\xi = c\,dt$$

$$u(x,t) = f(x - ct) + g(x + ct)$$

- Οι συναρτήσεις f, g προσδιορίζονται από τα αρχικά δεδομένα.

Παράδειγμα

$$u(x,0) = \sin(x), \ \frac{\partial u}{\partial t}(x,0) = -2c\cos(x)$$

$u(x,0) = f(x) + g(x) = \sin x$

$u_t(x,0) = -c f'(x) + c g'(x) = -2c \cdot \cos x$

$f(x) + g(x) = \sin x$

$f'(x) - g'(x) = 2\cos x$

$\downarrow$

$f(x) - g(x) = 2\sin x + c.$

$$f(x) = \tfrac{3}{2}\sin x \qquad g(x) = \sin x - \tfrac{3}{2}\sin x = -\frac{\sin x}{2}$$

$$u(x,t) = f(x-ct) + g(x+ct) =$$

$$= \tfrac{3}{2}\sin(x-ct) - \tfrac{1}{2}\sin(x+ct), \quad x \in \mathbb{R},\ t \in \mathbb{R}$$

Γενική περίπτωση

$$\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u(x,t)}{\partial t^2} = 0$$

$$u(x,0) = F(x), \quad \frac{\partial u(x,0)}{\partial t} = G(x)$$

Λύση

$$u(x,t) = \frac{F(x-ct)+F(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} G(\xi)d\xi$$

☻ Άσκηση: Κάντε την απόδειξη.

$$u_{xx} - c^{-2} u_{tt} = 0 \qquad u(x,0) = F(x) \qquad u_t(x,0) = G(x)$$

$$u(x,t) = f(x-ct) + g(x+ct)$$

$$u_t(x,t) = -c f'(x-ct) + c g'(x+ct)$$

$\underline{t=0}$:

$$f(x) + g(x) = F(x)$$

$$-cf'(x) + cg'(x) = G(x) \Leftrightarrow -f'(x) + g'(x) = \frac{1}{c} G(x)$$

$$\left.\begin{aligned} &\int_{-\infty}^{x} (-f'(\xi) + g'(\xi))\, d\xi = \frac{1}{c}\int_{-\infty}^{x} G(\xi)\, d\xi \\ &f(x) + g(x) = F(x) \end{aligned}\right\} \quad -f(x) + g(x) = \frac{1}{c}\int_{-\infty}^{x} G(\xi) d\xi + \text{const.}$$

$$2g(x) = F(x) + \frac{1}{c}\int_{-\infty}^{x} G(\xi)\,d\xi + \text{const.}$$

$$\boxed{g(x) = \frac{1}{2}F(x) + \frac{1}{2c}\int_{-\infty}^{x} G(\xi)\,d\xi + \text{const.}}$$

$$\boxed{f(x) = \frac{1}{2}F(x) - \frac{1}{2c}\int_{-\infty}^{x} G(\xi)\,d\xi - \text{const.}}$$

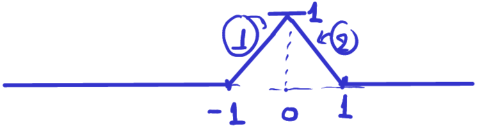
$$u(x,t) = f(x-ct) + g(x+ct) =$$

$$= \frac{1}{2}\Big(F(x-ct) + F(x+ct)\Big) + \frac{1}{2c}\int_{x-ct}^{x+ct} G(\xi)\,d\xi$$

$$u_{xx} - u_{tt} = 0$$

$$u(x,0) = F(x) = \begin{cases} -x+1, & x \in (0,1) \\ x+1, & x \in (-1,0] \\ 0 \end{cases}$$

$$u_t(x,0) = 0$$



$$\mathbb{1}\{x \in [0,1]\} = \begin{cases} 1, & x \in [0,1] \\ 0, & \text{drugače} \end{cases}$$

$$F(x) = (-x+1)\,\mathbb{1}\{x \in (0,1)\} + (x+1)\,\mathbb{1}\{x \in (-1,0]\}$$

$$u(x,t) = \frac{1}{2}\big(F(x-ct) + F(x+ct)\big) =$$

$$= \frac{1}{2}\Big((-x+ct+1)\,\mathbb{1}\{x \in (ct, ct+1)\} + (-x-ct+1)\cdot$$

$$\cdot\,\mathbb{1}\{x \in (-ct, -ct+1)\}$$

$$+ (x - ct + 1)\, \mathbb{1}\{x \in (-1+ct, ct]\} + (x + ct + 1) \cdot$$

$$\mathbb{1}\{x \in (-1-ct, -ct]\}. \Big)$$

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2η Διάλεξη - 18.2.2022

$\int_{-\infty}^{x} f'(\xi)\,d\xi = f(x) - \lim_{x\to -\infty} f(x)$

Μονοδιάστατη (1D) κυματική εξίσωση

$$\frac{\partial^2 u(x,t)}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u(x,t)}{\partial t^2} = 0$$

Αρμονικά κύματα

$\exp(ix) = \cos x + i\sin x$

$$u(x,t) = A\exp\{i(kx - \omega t)\}$$

Τυπικα: $u^*(x,t) = \mathrm{Re}\{u(x,t)\}$ $\quad A \in \mathbb{C} \to \exists\, r, \vartheta \in [0,2\pi)$ τ.ω. $A = r e^{i\theta}$

- Κυκλική συχνότητα (angular frequency): ω.
- Αριθμός κύματος (wavenumber): $k = \omega/c$.

Για κάθε k η $u(x,t)$ αποτελεί λύση της μονοδιάστατης κυματικής εξίσωσης για $\omega = ck$.

$u(x,t) = r e^{i\theta} e^{i(kx-\omega t)} = r e^{i(kx-\omega t+\vartheta)}$ $\quad \mathrm{Re}(u(x,t)) = r\cos(kx - \omega t + \vartheta)$

$$u_x = iku \quad , \quad u_{xx} = -k^2 u$$

$$u_t = -i\omega u \quad , \quad u_{tt} = -\omega^2 u$$

$$u_{xx} - c^{-2} u_{tt} = 0$$

$$-k^2 u + c^{-2}\omega^2 u = 0 \iff \frac{\omega^2}{c^2} = k^2 \quad \text{ή} \quad \omega^2 = c^2 k^2$$

$$\boxed{\omega = \pm ck}$$

$$\boxed{c = \frac{\omega}{k}} \leftarrow \text{Ταχύτητα.}$$

$$u_t + u_x + u_{xxx} = 0$$

- Αναζητούμε αρμονικές λύσεις

$$u(x,t) = \underbrace{A\exp\{i(kx-\omega t)\}}_{f(x-ct)} = A\exp\left\{ik\left(x - \frac{\omega}{k}t\right)\right\}$$

$$c = \frac{\omega}{k}$$

Σχέση διασποράς (dispersion relation): $\omega(k)$

- Δείχνει πως πρέπει να συνδέονται η συχνότητα και ο αριθμός κύματος ώστε να έχει μια εξίσωση αρμονική λύση.

$$u_t = -i\omega u \qquad u_x = iku \qquad u_{xxx} = -ik^3 u$$

$$-i\omega + ik - ik^3 = 0 \Rightarrow \omega(k) = k - k^3 \qquad c = \frac{\omega(k)}{k} = 1 - k^2$$

$$\frac{d\omega(k)}{dk} = 1 - 3k^2 = c_g \leftarrow \text{Ταχύτητα ομάδας.}$$

$$u_t + u_{xx} + u_{xxxx} = 0 \qquad u(x,t) = A\exp\{i(kx - wt)\} = A\exp\{i(kx - w(k)t)\}$$

$$u_t = -iw(k)\,u \qquad u_{xx} = (ik)^2 u = -k^2 u \qquad u_{xxxx} = (ik)^4 u = i k^4 u$$

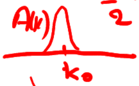
$$-iw(k) - k^2 + ik^4 = 0 \qquad C_g = \frac{dw(k)}{dk} = 2ik + 4k^3$$

$$\boxed{w(k) = ik^2 + k^4}$$

Έστω $k \in \left[k_0 - \frac{dk}{2}, k_0 + \frac{dk}{2}\right]$

$$\omega(k) = \omega(k_0) + (k-k_0)\frac{d\omega}{dk}(k_0) + O\left((k-k_0)^2\right)$$
$$= \omega_0 + (k-k_0)\,\omega_0'$$

Επιπλέον



$$u(x,t) = \int_{-\infty}^{\infty} A(k)\exp\{i(kx - \omega(k)t)\}dk = \int A(k)\exp\{i(kx \underbrace{+k_0x - k_0x}\; - \omega_0 t - (k-k_0)\omega_0' t)\}dk$$

$$= e^{i(k_0x - \omega_0 t)} \cdot \underbrace{\int A(k)\, e^{i(k-k_0)(x-\omega_0' t)}\, dk}_{f(x-\omega_0' t\,;\,k)}$$

- Σχηματισμός κυματικών πακέτων (wave packets)
- Ταξιδευουν με ταχύτητα ομάδας (group velocity) c_g

$$c_g = \frac{d\omega}{dk}$$

Εν γένει στα κυματικά φαινόμενα ισχύει

$$c_g < c = \frac{\omega}{k}$$

Παράδειγμα

$u_1(x,t) = \cos(x-t)$

$u_2(x,t) = \cos(2x-3t)$

$u(x,t) = u_1(x,t) + u_2(x,t)$

$\cos A + \cos B = 2\cos\frac{A-B}{2}\cos\frac{A+B}{2}$

$k_1, k_2 = ;$

$c = ;\quad c_g = ;$

Μετασχηματισμός Fourier

$$U(x,\omega) = \int_{-\infty}^{\infty} e^{\pm i\omega t} u(x,t)\,dt$$

$\int e^{i\omega t} u(x,t)\,dt$

Αντίστροφος μετασχηματισμός Fourier

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\mp i\omega t} U(x,\omega)\,d\omega$$

$\frac{1}{2\pi}\int e^{-i\omega t} U(x,\omega)\,d\omega$

Μετασχηματισμός Fourier

$$U(k, t) = \int_{-\infty}^{\infty} e^{\pm ikx} u(x, t) dx$$

Αντίστροφος μετασχηματισμός Fourier

$$u(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{\mp ikx} U(k, t) dk$$

$$(u_1 * u_2)(t)$$

$$u_1(t) * u_2(t) = \int_{-\infty}^{+\infty} u_1(\tau)\, u_2(t-\tau)\, d\tau$$

Ιδιότητες

Ιδιότητα	Συνάρτηση	Μετασχηματισμός Fourier
→ Πολλαπλασιασμός	$\underline{u_1(t)u_2(t)}$	$(2\pi)^{-1}U_1(\omega) * U_2(\omega)$
→ Συνέλιξη (convolution)	$u_1(t) * u_2(t)$	$U_1(\omega)U_2(\omega)$ ← Άσκηση.
Μεταφορά (translation)	$u(t-t_0)$	$e^{\pm i\omega t_0}U(\omega)$
Διαμόρφωση (modulation)	$e^{i\xi t}u(t)$	$U(\omega \pm \xi)$
Αλλαγή κλίμακας (scalling)	$u(t/s)$	$\mid s \mid U(s\omega)$
Παράγωγος (derivative)	$u^{(p)}(t)$	$(\mp i\omega)^p U(\omega)$

$$dt^* = dt$$

$$\int\!\!\int u_1(\tau)\, u_2(t-\tau)\, d\tau\, e^{i\omega t}\, dt = \int u_1(\tau) \int u_2(\underbrace{t-\tau}_{t^*})\, e^{i\omega t}\, dt\, d\tau =$$

$$= \int u_1(\tau)\, e^{i\omega\tau} d\tau \int u_2(t^*)\, e^{i\omega t^*} dt^* = U_1(\omega)\, U_2(\omega)$$

Παράδειγμα : $\delta^{(3)}(t) \xrightarrow[\mathcal{F}]{} (i\omega)^3 = -i\omega^3$ $\left(\delta(t) \xrightarrow[\mathcal{F}]{} 1\right)$

$$\int_{\mathbb{R}} \delta(t) e^{-i\omega t} dt = e^{-i\omega \cdot 0} = 1$$

$$u(x,t) = \int_{-\infty}^{\infty} A(k) \exp\{i(kx - \omega(k)t)\} dk$$

τι εκφράζει το $A(k)$?

$\underline{\underline{t=0}}$

$$u(x,0) = \int_{-\infty}^{+\infty} A(k) \exp(ikx) dk$$

$$u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} U(k,t) \exp(ikx) dk$$

χωρικός μετασχηματισμός Fourier.

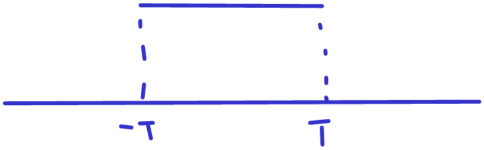
$$A(k) = \frac{1}{2\pi} U(k,0) = \mathcal{F}_{x \to k}\{u(x,0)\}$$

Παράδειγμα

Βρείτε τον μετασχηματισμό Fourier της δείκτριας συνάρτησης (indicator function)
$u(t) = 1_{[-T,T]}(t), \;\; T > 0$

$U(\omega) = ;$

$$U(\omega) = \int_{-\infty}^{+\infty} 1_{[-T,T]}(t)\, e^{i\omega t}\, dt = \int_{-T}^{T} e^{-i\omega t}\, dt = \dots \quad = -i\omega$$



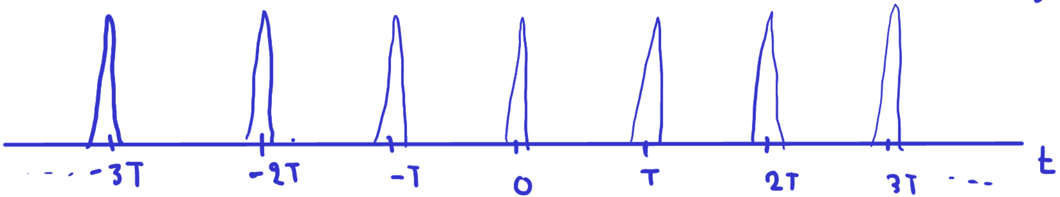
$$= \int_{-T}^{T} \frac{\left(e^{-i\omega t}\right)'}{-i\omega}\, dt$$

$$= -\frac{1}{i\omega}\left[e^{-i\omega t}\right]_{-T}^{T}$$

Παράδειγμα

Βρείτε τον μετασχηματισμό Fourier της $u(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT), \;\; T > 0.$

$U(\omega) = ;$



$\delta(t - nT)$ για κάποιο $n \in \mathbb{Z}$

$$\int_{-\infty}^{+\infty} \delta(t-nT)\, e^{i\omega t}\, dt = e^{i\omega nT}$$

$$\int \delta(x - x_0)\, f(x)\, dx = f(x_0)$$

άρα $\boxed{U(\omega) = \sum_{n=-\infty}^{\infty} e^{i\omega nT}}$

MEM-284: Κυματική Διάδοση

Τμήμα Μαθηματικών και Εφαρμοσμένων Μαθηματικών, Πανεπιστήμιο Κρήτης

Κώστας Σμαραγδάκης (https://kesmarag.gitlab.io)

3η Διάλεξη - 24.2.2022

$f(x,t)$

$$u_{xx} - c^{-2} u_{tt} + f = 0$$

$$u(x,0) = F(x), \quad u_t(x,0) = G(x)$$

Θα ορίσουμε 2 απλούστερα προβλήματα

Πρώτο πρόβλημα (Π1)

$$v_{xx} - c^{-2} v_{tt} = 0$$

$$v(x,0) = F(x), \quad v_t(x,0) = G(x)$$

Έχει γνωστή λύση (d'Alembert)

$$v(x,t) = \frac{F(x-ct) + F(x+ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} G(\xi)\, d\xi$$

Δεύτερο πρόβλημα (Π2)

$$w_{xx} - c^{-2} w_{tt} + f = 0$$

$$w(x,0) = 0, \quad w_t(x,0) = 0$$

Λήμμα

Εάν v, w οι λύσεις των Π1,Π2 αντίστοιχα, τότε η $u = v + w$ είναι η λύση της μη ομογενούς κυματικής εξίσωσης.

Αθροίζουμε τις εξισώσεις για v, w
και τις αρχικές συνθήκες.

Για την αντιμετώπιση του Π2 θα ορίσουμε άλλο ένα βοηθητικό πρόβλημα

Τρίτο πρόβλημα (Π3) με παράμετρο τ

$$q_{xx}(x,t;\tau) - c^{-2} q_{tt}(x,t;\tau) = 0,$$

$q(x,t;\tau)$

$\forall \tau$

$$q(x,\tau;\tau) = 0, \quad q_t(x,\tau;\tau) = c^2 f(x,\tau)$$

$q(x,t;t) = 0 \qquad q_t(x,t;t) = c^2 f(x,t)$

Λήμμα

Εάν $q(x,t;\tau)$ αποτελεί τη λύση του Π3 τότε

$w_x = \frac{\partial}{\partial x} w$

$$w(x,t) = \int_0^t q(x,t;\tau)\,d\tau$$

αποτελεί λύση του Π2

$$\frac{d}{dx}\left(\int_{\alpha(x)}^{b(x)} f(x,t)\,dt\right) = \int_{\alpha(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)\,dt + f(x,b(x))\,b'(x)$$

$$- f(x,\alpha(x)\,\alpha'(x)$$

$$w_{xx}(x,t) = \int_0^t q_{xx}(x,t;\tau)\,d\tau.$$

$$w_t(x,t) = \int_0^t q_t(x,t;\tau)\,d\tau + \overbrace{q(x,t;t)}^{0}$$

$$w_{tt}(x,t) = \int_0^t q_{tt}(x,t;\tau)\,d\tau + \overbrace{q_t(x,t;t)}^{c^2 f(x,t)}$$

$$= \int_0^t q_{tt}(x,t;\tau)\,d\tau + c^2 f(x,t)$$

$$c^{-2} w_{tt}(x,t) = \int_0^t c^{-2} q_{tt}(x,t;\tau)\,d\tau + f(x,t)$$

Π2 $\quad w_{xx} - c^{-2} w_{tt} + f = 0 \qquad w(x,0) = 0 \,, \quad w_t(x,0) = 0$

$$w_{xx} - c^{-2} w_{tt} = \int_0^t \underbrace{\left\{ q_{xx}(x,t;\tau) - c^{-2} q_{tt}(x,t;\tau) \right\}}_{\overset{\shortparallel}{0} \quad (\pi 3)} d\tau - f(x,t)$$

$$w(x,t) = \int_0^t q(x,t;\tau)\, d\tau \rightarrow w(x,0) = \int_0^0 \text{---}\, d\tau = 0$$

$$w_t(x,t) = \int_0^t q_t(x,t;\tau)\, d\tau \longrightarrow w_t(x,0) = \int_0^0 \text{---}\, d\tau = 0$$

Λύση του Π3

$$q_{xx}(x,t;\tau) - c^{-2} q_{tt}(x,t;\tau) = 0 \qquad t \geq \tau$$

$$q(x,\tau;\tau) = 0 \qquad q_t(x,\tau;\tau) = c^2 f(x,\tau)$$

$$t^* = t - \tau$$

$$q_{xx}(x,t^*) - c^{-2} q_{t^*t^*}(x,t^*) = 0$$

$$q(x,0) = 0 \qquad q_{t^*}(x,0) = c^2 f(x,0)$$

$$q(x,t;\tau) = \frac{c^{\cancel{2}}}{2\cancel{c}} \int_{x-c(t-\tau)}^{x+c(t-\tau)} f(\xi,\tau)\, d\xi .$$

$$u_{xx} - c^{-2}u_{tt} + f = 0$$

$$u(x,0) = F(x), \quad u_t(x,0) = G(x)$$

Λύση του προβλήματος

$$u(x,t) = \overbrace{\frac{F(x-ct)+F(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} G(\xi)d\xi}^{\Pi 1} + \overbrace{\frac{c}{2}\int_0^t \overbrace{\int_{x-c(t-\tau)}^{x+c(t-\tau)} f(\xi,\tau)d\xi}^{\Pi 3}\, d\tau}^{\Pi 2}$$

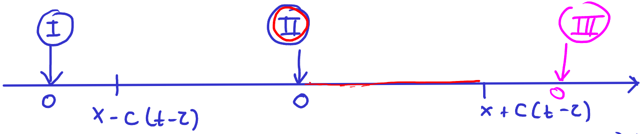
$$u_{xx} - c^{-2}u_{tt} + H(t)H(x) = 0$$

$$u(x,0) = \delta(x) \qquad u_t(x,0) = 0,$$

$$f(x,t) = H(t)H(x)$$

$$H(\xi) = \begin{cases} 1, & \xi > 0 \\ 0, & \text{otherwise.} \end{cases}$$

$$\int_{x-c(t-\tau)}^{x+c(t-\tau)} H(\tau)H(\xi)\,d\xi = H(\tau)\int_{x-c(t-\tau)}^{x+c(t-\tau)} H(\xi)\,d\xi.$$



I $\quad x - c(t-\tau) > 0 \Rightarrow x > c(t-\tau)$

$$\mathrm{I} = H(\tau)\int_{x-c(t-\tau)}^{x+c(t-\tau)} d\xi = 2H(\tau)c(t-\tau)$$

II $\quad x - c(t-\tau) < 0 < x + c(t-\tau)$

$-c(t-\tau) < x < c(t-\tau)$

$$\mathrm{II} = H(\tau)\int_{0}^{x+c(t-\tau)} d\xi = [x + c(t-\tau)]H(\tau)$$

III $\quad x + c(t-\tau) < 0 \Rightarrow x < -c(t-\tau)$

$$\mathrm{III} = 0$$

III $\int H(\tau)\cdot 0 \, d\tau = 0.$

II $\int_0^t [x + c(t-\tau)] \, d\tau = \quad xt + c\left(t^2 - \frac{t^2}{2}\right) = xt + \frac{c}{2} t^2$

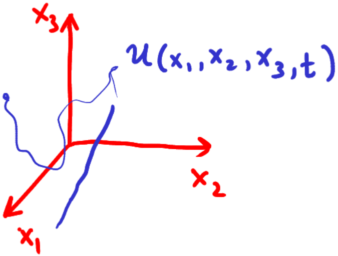
I ...

Κυματική εξίσωση στον χώρο (3D)

$\mathbb{R}^3$

Στη συνέχεια του μαθήματος θα ασχοληθούμε με προβλήματα τις μορφής

$$\mathbf{x} = (x_1, x_2, x_3), \quad u = u(\mathbf{x}, t), \quad f = f(\mathbf{x}, t), \quad \nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$$



$$\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right)$$

$$\nabla^2 = \nabla \cdot \nabla$$

$$\nabla^2 u - c^{-2} \frac{\partial^2 u}{\partial t^2} + f = 0,$$

$$u(\mathbf{x}; 0) = F(\mathbf{x}), \quad \frac{\partial}{\partial t} u(\mathbf{x}; 0) = G(\mathbf{x})$$

$$u_{x_1 x_1} + u_{x_2 x_2} + u_{x_3 x_3} - c^{-2} u_{tt} + f = 0$$

MEM-284: Κυματική Διάδοση

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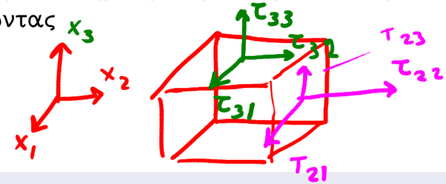
Κώστας Σμαραγδάκης (https://kesmarag.gitlab.io)

4η Διάλεξη - 25.2.2022

Δεν θα μελετήσουμε την παραγωγή της γραμμικοποιημένης κυματικής εξίσωσης. Αναφέρουμε όμως ότι προκύπτει χρησιμοποιώντας



- Διατήρηση μάζας
- Διατήρηση ορμής.
- Διατήρηση στροφορμής.

Συμβολισμοί

- $u_{,k} = \dfrac{\partial u}{\partial x_k}, \quad u_{i,k} = \dfrac{\partial u_i}{\partial x_k}$

$u_t \equiv \dot{u} \qquad u_{tt} \equiv \ddot{u}$

- $\dot{u} = \dfrac{\partial u}{\partial t}, \quad \ddot{u} = \dfrac{\partial^2 u}{\partial t^2}$

$\Sigma^T = \Sigma$

$$\Sigma = \begin{bmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{bmatrix}$$

Μεγέθη

- $\mathbf{u}$ - Διάνυσμα μετατοπίσεων. u_1, u_2, u_3
- τ_{ij} - Τάσεις (δύναμη ανά μονάδα επιφάνειας), ορίζουν (συμμετρικό) πίνακα τάσεων Σ.
- $\mathbf{f}$ - δύναμη πεδίου. $\underset{\sim}{f} = (f_1, f_2, f_3)$

Για ευκολία θα κάνουμε χρήση του συμβολισμού που εισήγαγε ο Albert Einstein για να γράφουμε τα αθροίσματα συνοπτικά.

- Εάν σε γινόμενα εμφανίζεται ένας δείκτης **ακριβώς 2 φορές** τότε υπάρχει άθροισμα για κάθε δυνατή τιμή του δείκτη.

Παράδειγμα - Εσωτερικό γινόμενο στις 3 διαστάσεις

$\alpha_{ii} = \alpha_{11} + \alpha_{22} + \alpha_{33}$

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 = \sum_{i=1}^{3} a_i b_i = a_i b_i$$

Παράδειγμα

$$a_{ij} b_j = \sum_{j=1}^{n} a_{ij} b_j$$

- Γραμμικοποιημένη κυματική εξίσωση για την διάδοση κυμάτων σε ελαστικό μέσο

$$\boxed{\tau_{ik,k} + f_i = \rho \ddot{u}_i}$$

Παραμόρφωση

$$\varepsilon = \left[u_{1,1} \quad \frac{u_{2,1} + u_{1,2}}{2} \quad \ldots \right.$$

$$\epsilon_{ij} = (\partial_i u_j + \partial_j u_i)/2$$

$$\partial_i u_j = \frac{\partial}{\partial x_i} u_j$$

Νόμος του Hook (για ομογενές και ισοτροπικό μέσο διάδοσης)

$$\delta_{ij} = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases}$$

$$\tau_{ij} = \lambda \epsilon_{kk} \delta_{ij} + 2\mu \epsilon_{ij}$$

$$= \lambda(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33})\delta_{ij} + 2\mu \varepsilon_{ij}$$

- λ, μ σταθερές ελαστικότητας Lame.

$$u_{1,i1} + u_{2,i2} + u_{3,i3}$$

$$(\lambda + \mu)u_{k,ik} + \mu u_{i,kk} + f_i = \rho \ddot{u}_i$$

$i=1$

$i=2$

$i=3$

$$\begin{bmatrix} \partial_{x_1} u_1 + \partial_{x_2} u_1 + \partial_{x_3} u_1 \\ \partial_{x_1} u_2 + \partial_{x_2} u_2 + \partial_{x_3} u_2 \\ \partial_{x_1} u_3 + \partial_{x_2} u_3 + \partial_{x_3} u_3 \end{bmatrix}$$

- Σε διανυσματική μορφή έχουμε

$$(\lambda + \mu)\nabla(\nabla \cdot \mathbf{u}) + \mu \nabla^2 \mathbf{u} + \mathbf{f} = \rho \ddot{\mathbf{u}}$$

- Χρησιμοποιώντας την ταυτότητα $\nabla^2 \mathbf{u} = \nabla(\nabla \cdot \mathbf{u}) - \nabla \times \nabla \times \mathbf{u}$ γράφουμε ισοδύναμα

$$\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right) \rightarrow (\lambda + 2\mu)\nabla(\underbrace{\nabla \cdot \mathbf{u}}_{\nabla^2 \varphi}) - \mu \nabla \times \nabla \times \mathbf{u} + \mathbf{f} = \rho \ddot{\mathbf{u}}$$

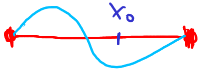
$$\underset{\sim}{u} = \nabla \varphi + \nabla \times \underset{\sim}{\psi}$$

$$\nabla \cdot \underset{\sim}{u} = (\nabla \cdot \nabla)\varphi = \nabla^2 \varphi$$

$$\nabla \ddot{\varphi} + \nabla \times \ddot{\underset{\sim}{\psi}}$$

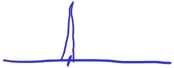
$$\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$$

$u'' + u + f(x) = 0 \qquad x \in [0, \pi/2] \qquad u(0) = u(\pi/2) = 0$



Συνάρτηση Green $\qquad u = G * f$

$G'' + G + \delta(x - x_0) = 0 \qquad G \doteq G(x; x_0)$



$x \neq x_0 \qquad G'' + G = 0 \qquad G(x;x_0) = Ae^{ix} + Be^{-ix}$

$$G'' = (i)^2 A e^{ix} + (-i)^2 B e^{-ix} = -G$$

$x < x_0$

$$G_I(x;x_0) = A_I e^{ix} + B_I e^{-ix}$$

$$G_I(0;x_0) = A_I + B_I = 0 \Rightarrow B_I = -A_I$$

$$\Big\} \quad G_I = A_I\left(e^{ix} - e^{-ix}\right) = 2iA_I \sin(x)$$

$x > x_0$

$$G_{II}(x;x_0) = A_{II} e^{ix} + B_{II} e^{-ix} \qquad e^{-ix} = \cos x - i\sin x$$

$$G_{II}(\pi/2; x_0) = 0 = iA_{II} - iB_{II} \Rightarrow A_{II} = B_{II}$$

$$G_{II} = A_{II}\left(e^{ix} + e^{-ix}\right) = 2A_{II}\cos(x)$$

$$G = \begin{cases} 2iA_I \sin(x), & x < x_0 \\ 2A_{II} \cos(x), & \underline{x > x_0} \end{cases} \qquad G' = \begin{cases} 2iA_I \cos(x) & x < x_0 \quad (x_0 - \varepsilon) \\ -2A_{II} \sin(x) & x > x_0 \quad (x_0 + \varepsilon) \end{cases}$$

$$x = x_0 \qquad \boxed{2iA_I \sin(x_0) = 2A_{II} \cos(x_0)} \qquad \varepsilon \to 0$$

$$G'' + G + \delta(x - x_0) = 0 \Rightarrow \int_{x_0-\varepsilon}^{x_0+\varepsilon} G'' + \underbrace{\int_{x_0-\varepsilon}^{x_0+\varepsilon} G}_{0} + \int_{x_0-\varepsilon}^{x_0+\varepsilon} \delta(x - x_0) =$$

$$= \left[G'\right]_{x_0-\varepsilon}^{x_0+\varepsilon} + 1 = 0$$

άρα $$\left[G'\right]_{x_0-\varepsilon}^{\underline{x_0+\varepsilon}} = -1$$

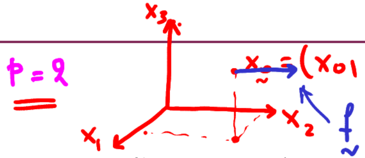
$$\varepsilon \to 0 \qquad -2A_{II} \sin(x_0 + \varepsilon) - 2iA_I \cos(x_0 - \varepsilon) = -1$$

$$\boxed{-2A_{II}\sin(x_0) - 2iA_I\cos(x_0) = -1}$$

$$\begin{bmatrix} 2i\sin(x_0) & -2\cos(x_0) \\ -2i\cos(x_0) & -2\sin(x_0) \end{bmatrix} \begin{bmatrix} A_I \\ A_{II} \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$f(x) = e^x$$

$$u(x) = e^{x_0} * g(x;x_0)$$



$\underset{\sim}{f}$: $f_i(\underset{\sim}{x}, t ; \underset{\sim}{x}_0)$

$\underset{\sim}{g} = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix}$

g_{i2} — g_{12}, g_{22}, g_{32}

Συμβολίζουμε ως $\mathcal{G}_{ip}(\mathbf{x}, t; \mathbf{x}_0)$ την i-συνιστώσα της λύσης της κυματικής εξίσωσης με την παρουσία της δύναμης $f_i(\mathbf{x}, t; \mathbf{x}_0) = \delta_{ip}\delta(\mathbf{x} - \mathbf{x}_0)\delta(t)$

$\delta(x_1 - x_{10})\,\delta(x_2 - x_{20})\,\delta(x_3 - x_{30})$

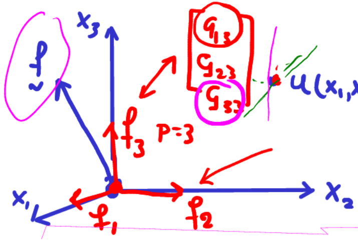
Θεώρημα αναπαράστασης

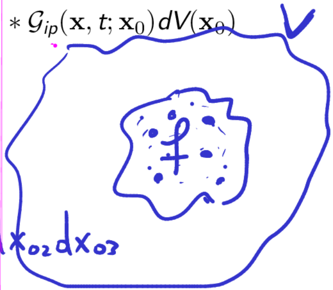
$$\mathbf{u}(\mathbf{x}, t) = \int_0^\infty d\tau \int_V f_p(\mathbf{x}, \tau)\mathcal{G}_{ip}(\mathbf{x}, t - \tau; \mathbf{x}_0)dV(\mathbf{x}_0) = \int_V f_p(\mathbf{x}, t) * \mathcal{G}_{ip}(\mathbf{x}, t; \mathbf{x}_0)dV(\mathbf{x}_0)$$

(*)

- Στο πεδίο συχνοτήτων $(t \to \omega)$

$$\mathbf{u}(\mathbf{x}, \omega) = \int_V f_p(\mathbf{x}, \omega)\mathcal{G}_{ip}(\mathbf{x}, \omega; \mathbf{x}_0)dV(\mathbf{x}_0)$$





$dx_{01}\, dx_{02}\, dx_{03}$

$(*) = \int_V \sum_{p=1}^{3} f_p * g_{ip}\, dV$

Δυναμικά μετατόπισης

$$\nabla \phi = \begin{bmatrix} \frac{\partial}{\partial x_1}\phi \\ \partial_{x_2}\phi \\ \partial_{x_3}\phi \end{bmatrix}$$

$$\mathbf{u} = \overbrace{\nabla \phi}^{\text{Παραμόρφωση}} + \overbrace{\nabla \times \boldsymbol{\psi}}^{\text{Στροφή.}}, \quad \nabla \cdot \boldsymbol{\psi} = 0$$

$$\ddot{u} = \nabla \ddot{\varphi} + \nabla \times \ddot{\underset{\sim}{\psi}}$$

$$\rho \ddot{u} = (\lambda + 2\mu) \nabla \left(\frac{\rho}{\lambda + 2\mu} \ddot{\varphi} \right) + \mu \nabla \times \left(\frac{\rho}{\mu} \ddot{\underset{\sim}{\psi}} \right)$$

Υποθέτουμε την περίπτωση χωρίς εξωτερική δύναμη πεδίου ($\underline{\mathbf{f} = \mathbf{0}}$)

$$\nabla \times \underset{\sim}{\psi} = \begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ \partial x_1 & \partial x_2 & \partial x_3 \\ \psi_1 & \psi_2 & \psi_3 \end{vmatrix}$$

$$(\lambda + 2\mu)\nabla\left[\nabla^2\phi - \alpha^{-2}\ddot{\phi}\right] + \mu\nabla \times \left[\nabla^2\boldsymbol{\psi} - \beta^{-2}\ddot{\boldsymbol{\psi}}\right] = \mathbf{0}$$

$$\partial_{x_1}^2 + \partial_{x_2}^2 + \partial_{x_3}^2$$

$$\partial_{x_1}\psi_1 + \partial_{x_2}\psi_2 + \partial_{x_3}\psi_3 = 0.$$

- Οδηγούμαστε στις κυματικές εξισώσεις για τα δυναμικά μετατόπισης

$$u_{xx} - c^{-2}\ddot{u} = 0$$

$$\phi_{x_1x_1} + \phi_{x_2x_2} + \phi_{x_3x_3} - \alpha^{-2}\ddot{\phi} = 0$$

$$\left.\begin{aligned} &\nabla^2\phi - \alpha^{-2}\ddot{\phi} = 0, \quad \alpha^2 = \frac{\lambda + 2\mu}{\rho} \\ &\nabla^2\boldsymbol{\psi} - \beta^{-2}\ddot{\boldsymbol{\psi}} = \mathbf{0}, \quad \beta^2 = \frac{\mu}{\rho} \end{aligned}\right\} \text{4 εξισώσεις.}$$

- Παρατηρούμε ότι $\alpha^2 > \beta^2$

Εξίσωση Poisson

$$\boxed{\nabla^2 u = f}$$

$$f = \delta(x - x_0)$$

$\cdot\, x_0$

$$\iiint_V \delta(\underset{\sim}{x} - \underset{\sim}{x_0})\, f(\underset{\sim}{x})\, dV = f(\underset{\sim}{x_0})$$

G

$$\delta(\mathbf{x} - \mathbf{x}_0) = \nabla^2 \left(\frac{-1}{4\pi \|\mathbf{x} - \mathbf{x}_0\|} \right)$$

Τι μπορούμε να πούμε για την παραπάνω σχέση?

$$\delta(\mathbf{x} - \mathbf{x}_0)$$

$$\|x - x_0\| = \left((x_1 - x_{01})^2 + (x_2 - x_{02})^2 + (x_3 - x_{03})^2 \right)^{1/2} = r$$

απόσταση του $\underset{\sim}{x}$ από το $\underset{\sim}{x_0}$

Δυναμικά μετατόπισης

Αντίστροφος Μετ. Fourier

λύσεις στο πεδίο συχνοτήτων.

$$\phi(\mathbf{x}, t) = \int \phi(\mathbf{x}, \omega) \exp(-i\omega t) d\omega$$

$\Phi(x,\omega)$

$$\psi(\mathbf{x}, t) = \int \psi(\mathbf{x}, \omega) \exp(-i\omega t) d\omega$$

Εξισώσεις Helmholz

$$\nabla^2 \phi(\mathbf{x}, \omega) + k_\alpha^2 \phi(\mathbf{x}, \omega) = 0, \quad k_\alpha^2 = \frac{\omega^2}{\alpha^2}$$

$$\nabla^2 \boldsymbol{\psi}(\mathbf{x}, \omega) + k_\beta^2 \boldsymbol{\psi}(\mathbf{x}, \omega) = \mathbf{0}, \quad k_\beta^2 = \frac{\omega^2}{\beta^2}$$

4 εξισώσεις.

$$\nabla^2 \varphi - \alpha^{-2} \ddot{\varphi} = 0 \qquad \varphi(x,t) = \int \Phi(x,\omega)\, e^{-i\omega t}\, d\omega$$

$$\nabla^2 \varphi = \int \nabla^2 \Phi\, e^{-i\omega t}\, d\omega$$

$$\ddot{\varphi} = \int \Phi\, (-i\omega)^2 e^{-i\omega t}\, d\omega = \int (-\omega^2)\, \Phi\, e^{-i\omega t}\, d\omega$$

$$\int \left[\nabla^2 \Phi + \frac{\omega^2}{\alpha^2} \Phi \right] e^{-i\omega t}\, d\omega = 0 \qquad [\omega] = rad/s$$

$$\nabla^2 \Phi + \frac{\omega^2}{\alpha^2} \Phi = 0 \qquad k_\alpha^2 = \left(\frac{\omega}{\alpha} \right)^2$$

ΜΕΜ-284: Κυματική Διάδοση

Τμήμα Μαθηματικών και Εφαρμοσμένων Μαθηματικών, Πανεπιστήμιο Κρήτης

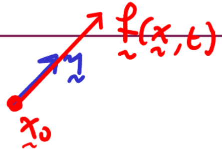
Κώστας Σμαραγδάκης (https://kesmarag.gitlab.io)

5η Διάλεξη - 3.2.2022

Εξίσωση Poisson

$$\nabla^2 \mathcal{G}(\mathbf{x}, t) = \delta(\mathbf{x})$$

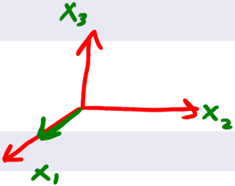
$$\mathcal{G}(\mathbf{x}, t) = -\frac{1}{4\pi r}, \quad r = [(x_1 - \underset{=0}{x_{01}})^2 + (x_2 - \underset{=0}{x_{02}})^2 + (x_3 - \underset{=0}{x_{03}})^2]^{1/2}$$



$$\mathbf{f}(\mathbf{x},t) = \mathbf{n}g(t)\delta(\mathbf{x}-\mathbf{x}_0)$$

Παράδειγμα : $\mathbf{n} = (1,0,0)$

$$\underset{\sim}{f} = \left(g(t)\,\delta(\underset{\sim}{x}-\underset{\sim}{x_0}),0,0\right)^T$$



$$\underset{\sim}{n} = (\delta_{1p},\delta_{2p},\delta_{3p})$$

Μια αναπαράσταση με δυναμικά για την f

$$\underset{\sim}{f} = \alpha^2\nabla F + \beta^2\nabla\times\mathbf{Z}$$

$$\alpha^2 F = -\nabla\cdot\left(\frac{\mathbf{n}}{4\pi r}\right)g(t)$$

$$\beta^2\mathbf{Z} = \nabla\times\left(\frac{\mathbf{n}}{4\pi r}\right)g(t)$$

$$\nabla^2\left(-\frac{1}{4\pi r}\right)=\delta(\underset{\sim}{x}) \qquad r=\|x\|$$

$$\nabla^2 \underset{\sim}{u} = \nabla(\nabla\cdot\underset{\sim}{u}) - \nabla\times\nabla\times\underset{\sim}{u}$$

$$\underset{\sim}{\eta}\,\delta(\underset{\sim}{x}) = \nabla\left(\nabla\cdot\left(-\frac{\underset{\sim}{\eta}}{4\pi r}\right)\right) - \nabla\times\nabla\times\left(-\frac{\underset{\sim}{\eta}}{4\pi r}\right)$$

$$\underset{\sim}{\eta}\,\delta(\underset{\sim}{x})g(t) = \nabla\Bigg(\underbrace{-\nabla\cdot\left(\frac{\underset{\sim}{\eta}}{4\pi r}\right)g(t)}_{\alpha^2 F}\Bigg) + \nabla\times\Bigg(\underbrace{\nabla\times\left(\frac{\underset{\sim}{\eta}}{4\pi r}\right)g(t)}_{\beta^2 \underset{\sim}{Z}}\Bigg)$$

$$= \alpha^2\nabla F + \beta^2\nabla\times\underset{\sim}{Z}.$$

$$\nabla^2\phi - \alpha^{-2}\ddot{\phi} - \alpha^{-2}g(t)\nabla\cdot\left(\frac{\mathbf{n}}{4\pi r}\right) = 0$$

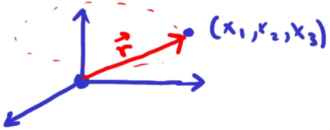
$$\nabla^2\boldsymbol{\psi} - \beta^{-2}\ddot{\boldsymbol{\psi}} + \beta^{-2}g(t)\nabla\times\left(\frac{\mathbf{n}}{4\pi r}\right) = \mathbf{0}$$

Βρίσκουμε Φ και Ψ βαθμωτά τ.ω.

- Ορίζουμε $\phi = \nabla\cdot(\Phi\mathbf{n})$, $\quad \boldsymbol{\psi} = -\nabla\times(\Psi\mathbf{n})$

$$\nabla^2(\nabla\cdot\Phi\underset{\sim}{n}) - \alpha^{-2}\,\nabla\cdot(\ddot{\Phi}\underset{\sim}{n}) - \alpha^{-2}g(t)\,\nabla\cdot\left(\frac{\underset{\sim}{n}}{4\pi r}\right) = 0$$

$\phi(x_1, x_2, x_3, t)$



$$\nabla^2 \Phi - \alpha^{-2}\ddot{\Phi} - \alpha^{-2} g(t) \frac{1}{4\pi r} = 0$$

$$\nabla^2 \Psi - \beta^{-2}\ddot{\Psi} - \beta^{-2} g(t) \frac{1}{4\pi r} = 0$$

- Οι δύο εξισώσεις διαφέρουν μόνο στις ταχύτητες διάδοσης (α, β).

$$\Phi(\mathbf{x}, t; \mathbf{x}_0)$$

3D

$$\nabla^2\Phi - \alpha^{-2}\ddot{\Phi} - \alpha^{-2}g(t)\frac{1}{4\pi r} = 0$$

Η εξίσωση στις σφαιρικές συντεταγμένες

Ασκηση ∇^2

$\Phi(\underset{\sim}{x}, t) \longrightarrow \phi(r,t)$

$$\Phi(r, t), \ \ r > 0$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial \Phi}{\partial r}) - \alpha^{-2}\ddot{\Phi} - \alpha^{-2}g(t)\frac{1}{4\pi r} = 0$$

$$\Phi(r, 0) = 0, \quad \dot{\Phi}(r, 0) = 0$$

← Αρχικές συνθήκες.

$$\frac{\partial \Phi}{\partial r} = \frac{r\mathcal{X}_r - \mathcal{X}}{r^2}$$

$$r^2 \phi_r = r\mathcal{X}_r - \mathcal{X}$$

$$(r^2\phi')' = \mathcal{X}_r + r\mathcal{X}_{rr} - \mathcal{X}_r = r\mathcal{X}_{rr}$$

- Το πρόβλημα λύνεται πιο εύκολα με αλλαγή μεταβλητής

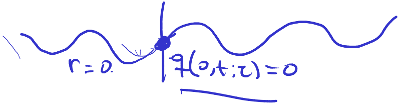
$$\frac{1}{r^2}(r^2\Phi')' = \frac{\mathcal{X}_{rr}}{r}$$

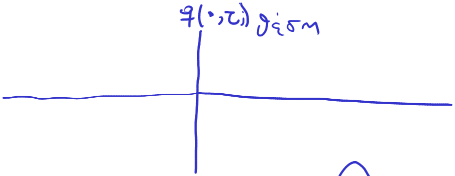
$$\ddot{\Phi} = \frac{\ddot{\mathcal{X}}}{r}$$

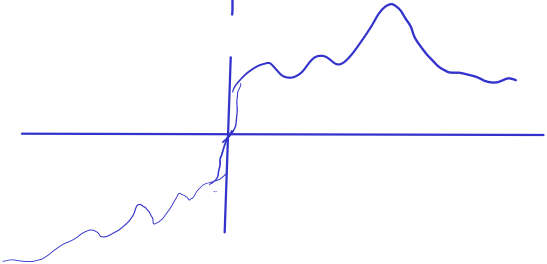
$$\Phi(r,t) = \frac{\mathcal{X}(r,t)}{r}, \quad \underline{r > 0}$$

$$\mathcal{X}_{rr} - \alpha^{-2}\ddot{\mathcal{X}} - \alpha^{-2}\frac{g(t)}{4\pi} = 0$$

$$\mathcal{X}(r,0) = 0, \quad \dot{\mathcal{X}}(r,0) = 0$$


r = 0
q(0,t;τ) = 0

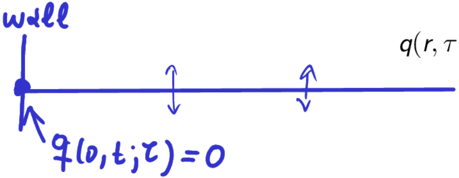

q(·,τ_i) given



$$\mathcal{X}_{rr} - \alpha^{-2}\ddot{\mathcal{X}} + f = 0$$

$$\mathcal{X}(r,0)=0, \quad \dot{\mathcal{X}}(r,0)=0.$$

$$q_{rr}(r,t;\tau) - \alpha^{-2}\ddot{q}(r,t;\tau) = 0$$



$$q(r,\tau;\tau) = 0, \quad \dot{q}(r,\tau;\tau) = -\frac{g(\tau)}{4\pi}$$

← Αρχικές συνθήκες (αρχή του χρόνου τ)

$$\mathcal{X}(r,t) = \int_0^t q(r,t;\tau)d\tau$$

- Το πρόβλημα είναι ορισμένο για $r > 0$
- Θα λύσουμε βρίσκοντας τη λύση για $r \in \mathbb{R}$ εισάγοντας επιπλέον τη συνοριακή συνθήκη $q(0,t;\tau) = 0$
- Στη συνέχεια θα περιορίσουμε τη λύση στο $r > 0$

Εάν τα αρχικά δεδομένα (αρχική μετατόπιση και ταχύτητα είναι περιττή συνάρτ.) τότε και η λύση θα είναι περιττή συνάρτηση ως προς το r

$H(r) = \begin{cases} 1, & r > 0 \\ 0, & r \leq 0 \end{cases}$

$f(-r) = -f(r)$

Q - συμβολίζει την επέκταση σε όλο το $\mathbb{R}$

$$Q_{rr}(r,t;\tau) - \alpha^{-2}\ddot{Q}(r,t;\tau) = 0$$

$r \in \mathbb{R}$

$f(r)H(r) - f(-r)H(-r)$

$$Q(r,\tau;\tau) = 0, \quad \dot{Q}(r,\tau;\tau) = -\frac{g(\tau)}{4\pi}H(r) + \frac{g(\tau)}{4\pi}H(-r) = \begin{cases} -\frac{g(\tau)}{4\pi}, & r > 0 \\ 0, & r = 0 \\ \frac{g(\tau)}{4\pi}, & r < 0 \end{cases}$$

Λυσή για $r > \alpha(t-\tau)$

$$Q(r,t;\tau) = -\frac{1}{2\alpha}\int_{r-\alpha(t-\tau)}^{r+\alpha(t-\tau)} \frac{g(\tau)}{4\pi}d\xi = -\frac{g(\tau)}{4\pi}(t-\tau)$$

Λυσή για $r < \alpha(t-\tau)$

$$Q(r,t;\tau) = -\frac{1}{2\alpha}\int_{\alpha(t-\tau)-r}^{r+\alpha(t-\tau)} \frac{g(\tau)}{4\pi}d\xi = -\frac{g(\tau)}{\alpha 4\pi}r$$

$$-\frac{1}{2\alpha}\int_{r-\alpha(t-\tau)}^{r+\alpha(t-\tau)} \frac{g(\tau)}{4\pi}\, d\xi = -\frac{1}{\cancel{2\alpha}}\frac{g(\tau)}{4\pi}\Big[\underbrace{\cancel{r}+\alpha(t-\tau)\cancel{-r}+\alpha(t-\tau)}_{\cancel{2\alpha}(t-\tau)}\Big]$$

$$-\frac{1}{2\alpha}\int_{\alpha(t-r)-r}^{r+\alpha(t-r)} \frac{g(\tau)}{4\pi}\, d\xi = -\frac{g(\tau)}{\cancel{2}\alpha\cdot 4\pi}\Big[\underbrace{r+\alpha(t-\tau)-\alpha(t-\tau)+r}_{\cancel{2}r}\Big]$$

$$q(r,t;\tau) = Q(r,t;\tau)\Big|_{r>0}$$

Λυσή για $r > \alpha(t-\tau)$

$$q(r,t;\tau) = -\frac{1}{2\alpha}\int_{r-\alpha(t-\tau)}^{r+\alpha(t-\tau)} \frac{g(\tau)}{4\pi}\,d\xi = -\frac{g(\tau)}{4\pi}(t-\tau)$$

Λυσή για $0 < r < \alpha(t-\tau)$

$$q(r,t;\tau) = -\frac{1}{2\alpha}\int_{\alpha(t-\tau)-r}^{r+\alpha(t-\tau)} \frac{g(\tau)}{4\pi}\,d\xi = -\frac{g(\tau)}{\alpha 4\pi}r$$

$t-\tau-r/\alpha>0$

$\alpha(t-\tau)-r>0$

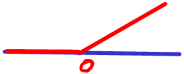
- Συνοπτικά γράφουμε

$$q(r,t;\tau) = \left(\frac{R(t-\tau-r/\alpha)}{4\pi} - \frac{t-\tau}{4\pi}\right)g(\tau)$$

$$\frac{\cancel{t}-\cancel{\tau}-r/\alpha-\cancel{t}+\cancel{\tau}}{4\pi}$$

$t-\tau-r/\alpha<0$

$$-\frac{t-\tau}{4\pi}g(\tau)$$



όπου $R(\xi) = \xi H(\xi) = \begin{cases} \xi, & \xi>0 \\ 0, & \xi\le 0 \end{cases}$

$$\chi(r,t) = \int_0^t \left(\frac{R(t-\tau-r/\alpha)}{4\pi} - \frac{t-\tau}{4\pi} \right) \delta(\tau)\, d\tau$$

$\tau=0$

$$= \frac{R(t-r/\alpha)}{4\pi} - \frac{t}{4\pi}.$$

$$\int f(t)\delta(t-t_0)dt = f(t_0)$$

$$\Phi = \frac{\chi}{r}$$

Για $g(t) = \delta(t)$ έχουμε

$$\xi(r,t) = \frac{R(t-r/\alpha)}{4\pi} - \frac{t}{4\pi}$$

Επομένως

$$\boxed{\Phi(r,t) = \frac{R(t-r/\alpha)}{4\pi r} - \frac{t}{4\pi r}, \quad r > 0}$$

Ομοία

$$\boxed{\Psi(r,t) = \frac{R(t-r/\beta)}{4\pi r} - \frac{t}{4\pi r}, \quad r > 0}$$

$\varphi = \nabla \cdot (\Phi \underset{\sim}{n})$ $\quad \underset{\sim}{\psi} = -\nabla \times (\Psi \underset{\sim}{n})$ $\quad H(t) = \begin{cases} 1, & t>0 \\ 0, & t \leq 0 \end{cases}$

$G_{ip} = \nabla \varphi + \nabla \times \underset{\sim}{\psi}$

$R(t) = tH(t)$

$G_{ip}(\underset{\sim}{x}, t)$

$$\mathcal{G}_{ip} = \frac{\gamma_i\gamma_p}{4\pi\rho\alpha^2 r}\delta(t - r/\alpha) + \frac{(\delta_{ip} - \gamma_i\gamma_p)}{4\pi\rho\beta^2 r}\delta(t - r/\beta)$$

$$(*)\begin{cases} - \dfrac{(\delta_{ip} - 3\gamma_i\gamma_p)}{4\pi\rho\alpha^2}\left[\dfrac{\alpha}{r^2}H(t - r/\alpha) + \dfrac{\alpha^2}{r^3}R(t - r/\alpha)\right] \\ + \dfrac{(\delta_{ip} - 3\gamma_i\gamma_p)}{4\pi\rho\beta^2}\left[\dfrac{\beta}{r^2}H(t - r/\beta) + \dfrac{\beta^2}{r^3}R(t - r/\beta)\right] \end{cases}$$

$A = \dfrac{\delta_{ip} - 3\gamma_i\gamma_p}{4\pi\rho}$

όπου $\gamma_i = x_i/r$

$t > r/\alpha$.

$$\frac{\alpha}{r^2}H(t - r/\alpha) + \frac{\alpha^2}{r^3}R(t - r/\alpha) = \frac{\alpha}{r^2} + \frac{\alpha^2}{r^3}(t - r/\alpha) = \frac{\alpha^2}{r^3}t$$

$$= \begin{cases} \dfrac{\alpha}{r^2} + \dfrac{\alpha^2}{r^3}\cdot(t - r/\alpha) = \dfrac{\alpha}{r^2} + \dfrac{\alpha^2}{r^3}t - \dfrac{\alpha}{r^2} = \dfrac{\alpha^2}{r^3}t, & t > r/\alpha \\ 0, & \text{διαφορετικά} \end{cases}$$

$$t > r/\alpha \qquad\qquad t > r/\beta$$

$$(*) \quad -A \cdot \frac{t}{r^3} + A\frac{t}{r^3}$$

$$-A\frac{\cancel{r}}{r^{\cancel{3}}\alpha} = -\frac{A}{r^2\alpha}$$

$-A/r^2\alpha$

Τελικά η συνάρτηση Green γράφεται

$$\mathcal{G}_{ip}(\mathbf{x}, t) = \frac{\gamma_i\gamma_p}{4\pi\rho\alpha^2 r}\delta(t - r/\alpha) + \frac{(\delta_{ip} - \gamma_i\gamma_p)}{4\pi\rho\beta^2 r}\delta(t - r/\beta)$$
$$- \frac{(\delta_{ip} - 3\gamma_i\gamma_p)}{4\pi\rho r^3} tH(t, r/\alpha, r/\beta)$$

$-A/r^2\beta$

$$\delta_{ip} = \begin{cases} 1, & i = p \\ 0, & i \neq p \end{cases}$$

$$H(t, r/\alpha, r/\beta) = \begin{cases} 1, & t \in [r/\alpha, r/\beta) \\ 0, & \text{διαφορετικά} \end{cases}$$

$\delta(\underset{\sim}{x}-\underset{\sim}{x}_0)=\delta(x_1-x_{01})\delta(x_2-x_{02})\delta(x_3-x_{03})$

$r = 10.000\ m$

Λύση μακρινού πεδίου (far field) $r \gg 0$

$\int_{\mathbb{R}^3}\delta(\underset{\sim}{x}-\underset{\sim}{x}_0)f(\underset{\sim}{x})\,dV = f(\underset{\sim}{x}_0)$

$$\mathcal{G}_{ip} = \mathcal{G}_{ip}^P + \mathcal{G}_{ip}^S$$

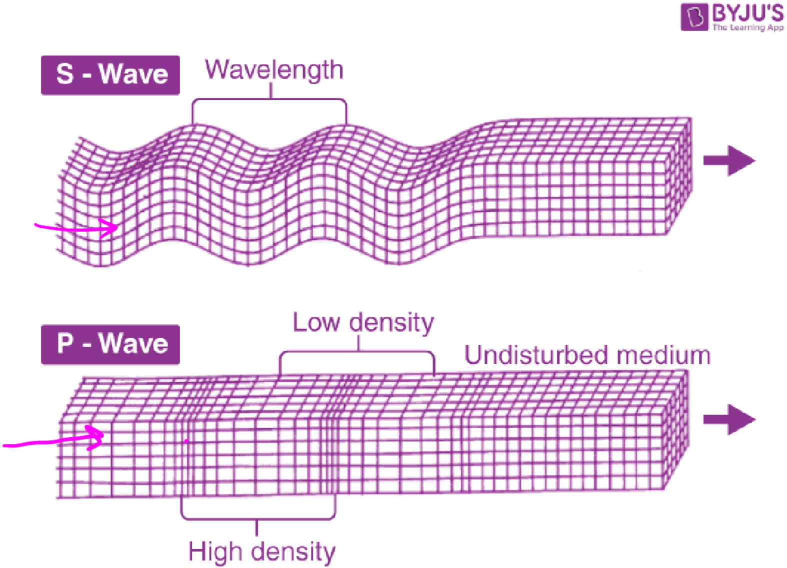
P-wave Primary

$\gamma_i = \frac{x_i}{r}$

$$\mathcal{G}_{ip}^P = \frac{\gamma_i\gamma_p}{4\pi\rho\alpha^2 r}\delta(t - r/\alpha)$$

S-wave Secondary

$$\mathcal{G}_{ip}^S = \frac{(\delta_{ip} - \gamma_i\gamma_p)}{4\pi\rho\beta^2 r}\delta(t - r/\beta)$$


BYJU'S
The Learning App
S - Wave
Wavelength
P - Wave
Low density
Undisturbed medium
High density

MEM-284: Κυματική Διάδοση

Τμήμα Μαθηματικών και Εφαρμοσμένων Μαθηματικών, Πανεπιστήμιο Κρήτης

Κώστας Σμαραγδάκης (https://kesmarag.gitlab.io)

6η Διάλεξη - 10.3.2022

① $\delta(t)\delta(\underset{\sim}{x})\hat{e}_p$ η λύση είναι η $\mathcal{G}_{ip}(\underset{\sim}{x},t)$

$p \in \{1,2,3\}$

$x_1 \quad x_2 \quad x_3$

$\uparrow q$

$$\begin{bmatrix} \mathcal{G}_{1p} \\ \mathcal{G}_{2p} \\ \mathcal{G}_{3p} \end{bmatrix}$$

$\delta(t)\delta(\underset{\sim}{x})$

r

$\hat{e}_3$

$\mathcal{G}_{i3}(\underset{\sim}{x},t)$

$(0,0,0)$

② $\underset{\sim}{f}(\underset{\sim}{x},t)$ → υπολογισμός της λύσης $\underset{\sim}{u}(\underset{\sim}{x},t)$

$$\mathbf{u}(\mathbf{x},t) = \int_0^\infty d\tau \int_V f_p(\mathbf{x}_0,\tau)\mathcal{G}_{ip}(\mathbf{x},t-\tau;\mathbf{x}_0)dV(\mathbf{x}_0) = \int_V f_p(\mathbf{x}_0,t) * \mathcal{G}_{ip}(\mathbf{x},t;\mathbf{x}_0)dV(\mathbf{x}_0)$$

- Στο πεδίο συχνοτήτων $(t \to \omega)$

$$\mathbf{u}(\mathbf{x},\omega) = \int_V f_p(\mathbf{x}_0,\omega)\mathcal{G}_{ip}(\mathbf{x},\omega;\mathbf{x}_0)dV(\mathbf{x}_0)$$

①b $\delta(t)\delta(\underset{\sim}{x}-\underset{\sim}{x}_0)\hat{e}_p$

r

$\delta(t)\delta(\underset{\sim}{x}-\underset{\sim}{x}_0)$

$\hat{e}_3$

$\mathcal{G}_{i3}(\underset{\sim}{x},t;\underset{\sim}{x}_0)$

(x_{01},x_{02},x_{03})

$r \gg 1$

$$\mathcal{G}_{ip} = \mathcal{G}_{ip}^{P} + \mathcal{G}_{ip}^{S}$$

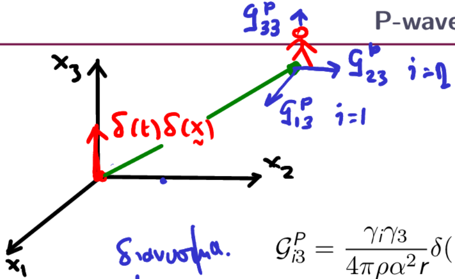
P-wave

$$\mathcal{G}_{ip}^{P} = \frac{\gamma_i \gamma_p}{4\pi\rho\alpha^2 r}\delta(t - r/\alpha)$$

$\gamma_i = \frac{x_i}{r} \in [-1, 1]$ $\quad r = (x_1^2 + x_2^2 + x_3^2)^{1/2}$

S-wave

$$\mathcal{G}_{ip}^{S} = \frac{(\delta_{ip} - \gamma_i \gamma_p)}{4\pi\rho\beta^2 r}\delta(t - r/\beta)$$



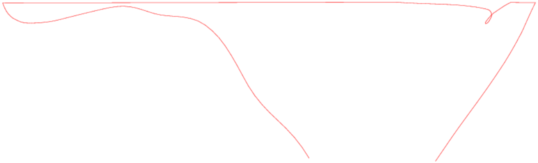
$\alpha = 5000$ m/s

$r = 10000$ m

διάνυσμα.

$$\mathcal{G}^P_{i3} = \frac{\gamma_i \gamma_3}{4\pi\rho\alpha^2 r}\delta(t - r/\alpha) = \frac{\gamma_3}{4\pi\rho\alpha^2}\ \gamma_i\ \frac{1}{r}\ \delta(t - r/\alpha)$$

- $\dfrac{\gamma_3}{4\pi\rho\alpha^2}\gamma$: Πρότυπο ακτινοβολίας (radiation pattern). Κατευθυντικότητα
- $\dfrac{1}{r}$: Εξασθένιση με την απόσταση.
- $\delta(t - r/\alpha)$: Οδευών παλμός που απομακρύνεται με ταχύτητα α.



$\gamma_3 \underset{\sim}{\gamma}$

$\gamma_3 = \dfrac{x_3}{r}, \quad \underset{\sim}{\gamma} = (\gamma_1, \gamma_2, \gamma_3)$

$\theta \in [0, 2\pi)$

$\varphi \in [0, \pi]$

$x_1 = r \sin\varphi \cos\theta$

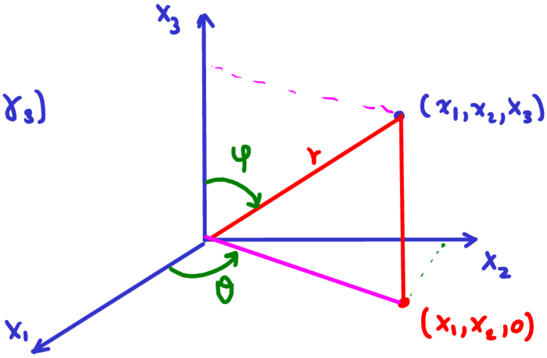
$x_2 = r \sin\varphi \sin\theta$

$x_3 = r \cos\varphi$

$\gamma_1 = \dfrac{x_1}{r} = \sin\varphi \cos\theta$

$\gamma_2 = \sin\varphi \sin\theta$

$\gamma_3 = \cos\varphi$



$(x_1, x_2, x_3) \leftrightarrow (r, \theta, \varphi)$

Μας ενδιαφέρει το μέτρο του $\gamma_3 \underset{\sim}{\gamma}$. Μέτρο μετατόπισης λόγω προσανατολισμο…

$$\|\gamma_3 \underset{\sim}{\gamma}\| = |\gamma_3| \|\underset{\sim}{\gamma}\| = |\cos\varphi| \cdot \|\underset{\sim}{\gamma}\|$$

$$\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = \|\underset{\sim}{\gamma}\|^2 = \sin^2\varphi\cos^2\theta + \sin^2\varphi\sin^2\theta + \cos^2\varphi =$$

$$= \sin^2\varphi\,(\underbrace{\cos^2\theta + \sin^2\theta}_{=1}) + \cos^2\varphi =$$

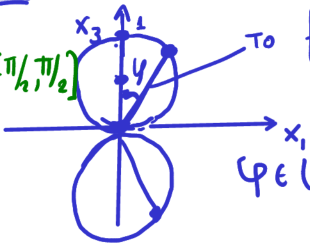
$$= \sin^2\varphi + \cos^2\varphi = 1 \qquad \text{άρα } \|\underset{\sim}{\gamma}\| = 1$$

$\|\gamma_3 \underset{\sim}{\gamma}\| = |\cos\varphi|$ ανεξάρτητο του θ.

P-wave Πρότυπο ακτινοβολίας

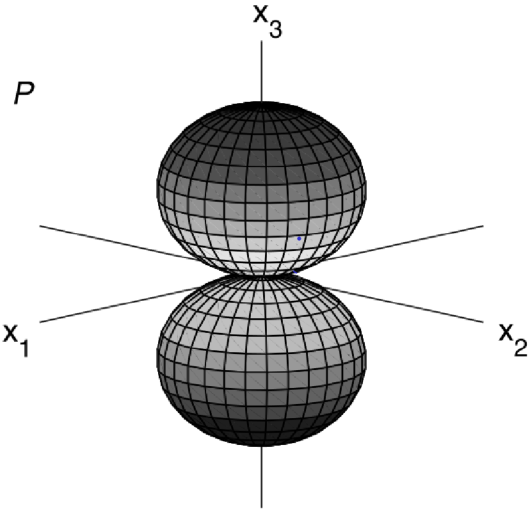
$\underline{\theta = 0}$ $\quad - x_1 x_3 -$ $\quad \propto |\cos\varphi|$ $\quad \varphi \in [0, \pi]$ $\quad x_3 = r\cos\varphi$

$\varphi \in [-\pi/2, \pi/2]$

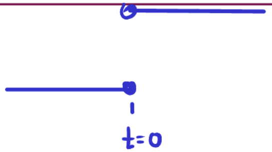


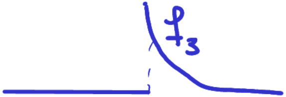
Το μήκος δηλώνει πόσο έντονο είναι το πεδίο λόγω της κατευθυντικότητας.

$\varphi \in (\pi/2, 3\pi/2)$


x3
P
x1
x2

$$H(t) = \begin{cases} 1, & t>0 \\ 0, & t\leq 0 \end{cases}$$



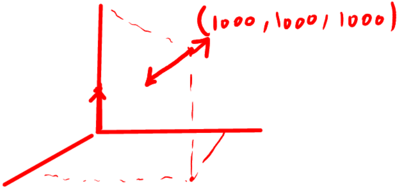


$f_1 = f_2 = 0$

Διάδοση p-wave στις 3 διαστάσεις

$$\mathbf{f} = (0, 0, \overbrace{H(t)e^{-t}\delta(\mathbf{x})}^{f_3})$$

- Απείρο μέσο με πυκνότητα $2500kg/m^3$ και ταχύτητα διάδοσης για p-wave $5600m/s$.
- Εφαρμογή με python.



$$G_{i3} = \frac{\gamma_3 \gamma_i}{4\pi\rho\alpha^2} \frac{1}{r} \cdot \delta(t - r/\alpha) \qquad f_3(\underset{\sim}{x}, t) = H(t) e^{-t} \delta(\underset{\sim}{x})$$

$$u_i(\underset{\sim}{x}, t) = \int_0^t d\tau \int_V f_3(\underset{\sim}{x}_0, \tau)\, G_{i3}(\underset{\sim}{x}, t-\tau; \underset{\sim}{x}_0)\, dV(\underset{\sim}{x}_0) =$$

$$= \int_0^t d\tau \int_V H(\tau) e^{-\tau} \delta(\underset{\sim}{x}_0)\, G_{i3}(\underset{\sim}{x}, t-\tau; x_0)\, dV(\underset{\sim}{x}_0) =$$

$$= \int_0^t H(\tau) e^{-\tau} G_{i3}(\underset{\sim}{x}, t-\tau)\, d\tau =$$

$t - \tau - r/\alpha = 0 \Leftrightarrow \tau = t - r/\alpha$

$$= \frac{\gamma_3 \gamma_j}{4\pi\rho\alpha^2 r} \int_0^t H(\tau) e^{-\tau} \delta(t - \tau - r/\alpha)\, d\tau =$$

$$= \frac{\gamma_3 \gamma_j}{4\pi\rho\alpha^2 r} H(t - r/\alpha) e^{-(t - r/\alpha)}$$

$$\left.\begin{array}{l} (x_1, x_2, x_3) \text{ 1 σημείο} \\ t \leftarrow \text{διάνυσμα με χρονικές στιγμές.} \\ \quad\text{με διάσταση } d \end{array}\right\} \rightarrow \underset{\sim}{u} \in \mathbb{R}^{3,d}$$

```
   0  1  2
L=[3, 5, 8]

for i,e in enumerate(L):
    print(i,e)
```


```
i=0
for e in L:
    print(i,e)
    i+=1
```

MEM-284: Κυματική Διάδοση

Τμήμα Μαθηματικών και Εφαρμοσμένων Μαθηματικών, Πανεπιστήμιο Κρήτης

Κώστας Σμαραγδάκης (https://kesmarag.gitlab.io)

1ο εργαστήριο ασκήσεων - 11.3.2022

$$\int_{-\infty}^{+\infty} f(x)\,\delta(x-x_0)\,dx = f(x_0)$$

$$H(x) = \begin{cases} 1, & x>0 \\ 0, & x\le 0 \end{cases}$$

Λύστε την παρακάτω κυματική εξίσωση

$$u_{xx} - \underbrace{\frac{1}{4}}_{c^{-2}} u_{tt} + \overbrace{\delta(x+3)(\delta(t)+\delta(t-1))}^{f(x,t)} = 0, \quad x\in\mathbb{R}, t\in\mathbb{R}$$

$$u(x,0) = \overset{F(x)}{\sin(x)}, \quad u_t(x,0) = \overset{G(x)}{H(x)}$$

$$u(x,t) = \overset{1}{\frac{F(x-ct)+F(x+ct)}{2}} + \underset{2}{\frac{1}{2c}\int_{x-ct}^{x+ct} G(\xi)\,d\xi} + \overset{3}{\frac{c}{2}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} f(\xi,\tau)\,d\xi\,d\tau}$$

1 $\quad \dfrac{\sin(x-2t)+\sin(x+2t)}{2}$

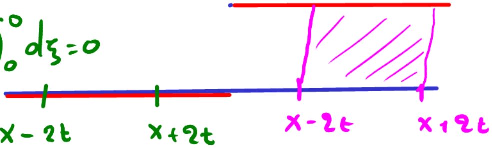
2 $\quad \dfrac{1}{4}\int_{x-2t}^{x+2t} H(\xi)\,d\xi =$

$\max(0, x+2t)$

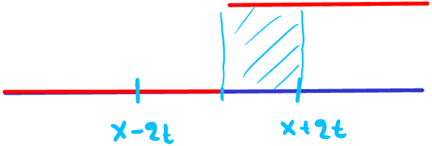
$\max(0, x-2t)$

$$= \frac{1}{4}\int_{\max(0,\,x-2t)}^{\max(0,\,x+2t)} d\xi = \frac{1}{4}\left[\max(0,x+2t) - \max(0,x-2t)\right]$$

$\int_0^0 d\xi = 0$



$x+2t-(x-2t)=4t$



$$f(x,t) = \delta(x+3)\left(\delta(t)+\delta(t-1)\right)$$

$$\int_{x-2(t-\tau)}^{x+2(t-\tau)} \delta(\xi+3)\left(\delta(\tau)+\delta(\tau-1)\right) d\xi =$$

$$\int_{-\infty}^{+\infty} \delta(\xi+3)\, d\xi = 1$$

$$= \left(\delta(\tau)+\delta(\tau-1)\right) \underbrace{\int_{x-2(t-\tau)}^{x+2(t-\tau)} \delta(\xi+3)\, d\xi}_{\mathbb{1}\left\{-3 \in \left[x-2(t-\tau),\, x+2(t-\tau)\right]\right\}}$$

$$\mathbb{1}\{\text{συνθήκη}\} = \begin{cases} 1, & \text{συνθήκη ισχύει} \\ 0, & \text{διαφορετικά} \end{cases}$$

$$\int_0^t \left(\delta(\tau)+\delta(\tau-1)\right) \mathbb{1}\left\{-3 \in \left[x-2(t-\tau),\, x+2(t-\tau)\right]\right\} d\tau =$$

$$\int_0^t \delta(\tau)\, \mathbb{1}\left\{-3 \in \left[\overset{\tau}{\cdots}\right]\right\} d\tau + \int_0^t \delta(\tau-1)\, \mathbb{1}\left\{-3 \in \left[\overset{\tau}{\cdots}\right]\right\} d\tau =$$

$$= \mathbb{1}\left\{-3 \in \left[x-2t,\, x+2t\right]\right\} + \mathbb{1}\left\{-3 \in \left[x-2t+2,\, x+2t-2\right]\right\}$$

Θεωρώντας αρμονικές λύσεις, λύστε την εξίσωση

$$u_t + u_x - u_{xx} = 0 \qquad \omega(k)$$

$$u = Ae^{i(kx-wt)}$$

$$k^2 + ik - i\omega = 0$$

$$\frac{k^2}{i} = \frac{ik^2}{i^2}$$

$$i\omega = k^2 + ik$$

$$\omega(k) = \frac{k^2}{i} + k = k - ik^2$$

$$u(x,t) = Ae^{i(kx-kt)-k^2} = Ae^{-k^2 t}e^{i(kx-kt)}$$

μοναδιαία ταχύτητα $c = \frac{k}{k} = 1.$

Άσκηση 2

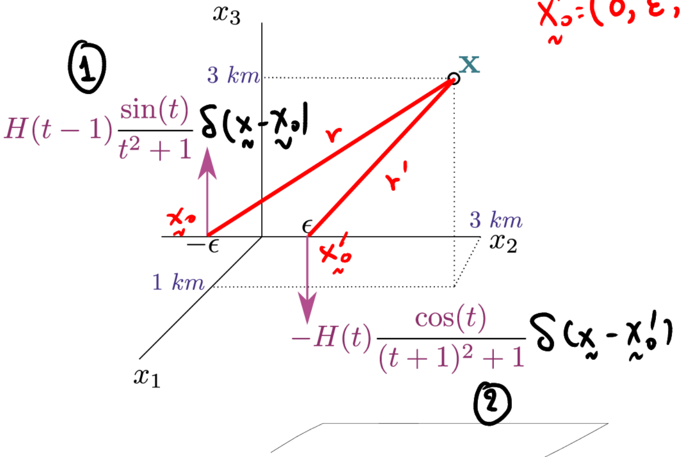
Υπολογίστε την μετατόπιση στο σημείο x για $t > 0$

$$\rho = 2500\ kg/m^3,\ \alpha = 6000\ m/s$$

Σημεία

$\underset{\sim}{x}_0 = (0, -\varepsilon, 0)$

$\underset{\sim}{x}_0' = (0, \varepsilon, 0)$



$$u_i(\underset{\sim}{x},t) = \int_0^t d\tau \underbrace{\int_V f_3(\underset{\sim}{\xi},\tau)\, G_{i3}(\underset{\sim}{x}, t-\tau; \underset{\sim}{\xi})\, dV(\underset{\sim}{\xi})}_{I}$$

1

$$I = H(\tau-1)\frac{\sin\tau}{\tau^2+1}\int_V \delta(\underset{\sim}{\xi}-\underset{\sim}{x_0})\, G_{i3}(\underset{\sim}{x}, t-\tau); \underset{\sim}{\xi})\, dV(\underset{\sim}{\xi}) =$$

$$= H(\tau-1)\frac{\sin\tau}{\tau^2+1}\, G_{i3}(\underset{\sim}{x}, t-\tau; \underset{\sim}{x_0})$$

$$G_{i3}(\underset{\sim}{x}, t) = \frac{\gamma_i\gamma_3}{4\pi\rho\alpha^2}\frac{1}{r}\delta(t-r/\alpha)$$

$$G_{i3}(\underset{\sim}{x}, t; \underset{\sim}{x_0}) = \frac{\gamma_i \gamma_3}{4\pi\rho\alpha^2} \frac{1}{r} \delta(t - r/\alpha)$$

$$\gamma_i = \frac{x_i - x_{0i}}{r}$$

$$r = \| \underset{\sim}{x} - \underline{x}_0 \|$$

σταθερά ως προς τ

$$\int_0^t \frac{\gamma_i \gamma_3}{4\pi\rho\alpha^2} \frac{1}{r} \delta(t - \tau - r/\alpha) H(\tau - 1) \frac{\sin \tau}{\tau^2 + 1} d\tau =$$

$$= \frac{\gamma_i \gamma_3}{4\pi\rho\alpha^2} \frac{1}{r} H\left(t - r/\alpha - 1\right) \frac{\sin(t - r/\alpha)}{(t - r/\alpha)^2 + 1}$$

Άσκηση 3

Δείξτε ότι $(f * g)(t) = (g * f)(t)$ θ.ν.δ.ο $\tau^* = t - \tau \to \tau = t - \tau^*$

$$\int_{-\infty}^{+\infty} f(\tau)\, g(\overbrace{t-\tau}^{\tau^*})\, d\tau = \int_{-\infty}^{+\infty} g(\tau)\, f(t-\tau)\, d\tau$$

$$d\tau^* = -d\tau$$

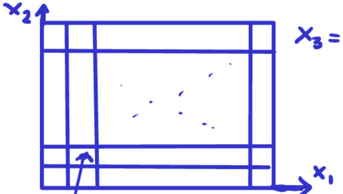
$$\int_{+\infty}^{-\infty} f(t-\tau^*)\, g(\tau^*)\, (-d\tau^*) = \int_{-\infty}^{+\infty} g(\tau^*)\, f(t-\tau^*)\, d\tau^* = $$

$$= g * f$$

$$f * g \qquad F(\omega)G(\omega) = G(\omega)F(\omega) \longrightarrow g * f$$

Άσκηση 4

Ασκ 3
Φυλ. 1



$x_3 = 3\,km$

plt.pcolor

subplot

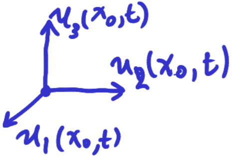
MEM-284: Κυματική Διάδοση

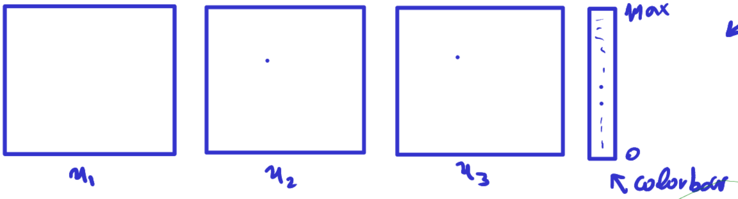
Τμήμα Μαθηματικών και Εφαρμοσμένων Μαθηματικών, Πανεπιστήμιο Κρήτης

u_1 μετατοπιση στην κατευθυνση του x_1
u_2 » x_2
u_3 x_3

Κώστας Σμαραγδάκης (https://kesmarag.gitlab.io)

7η Διάλεξη - 17.3.2022



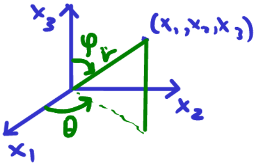


$\delta(\underset{\sim}{x}-\underset{\sim}{x_0})\,\delta(t)$

$i=1,2,3$

$$\delta_{ij} = \begin{cases} 1, & i=j \\ 0, & i\neq j \end{cases}$$

$$\mathcal{G}^S_{i3} = \frac{\delta_{i3} - \gamma_i\gamma_3}{4\pi\rho\beta^2 r}\delta(t - r/\beta)$$



$$\begin{bmatrix} g_{13} \\ g_{23} \\ g_{33} \end{bmatrix} =$$

- $\dfrac{1}{4\pi\rho\beta^2}[-\gamma_1\gamma_3,\ -\gamma_2\gamma_3,\ 1-\gamma_3^2]^T$: Πρότυπο ακτινοβολίας (radiation pattern).
- $\dfrac{1}{r}$: Εξασθένιση με την απόσταση.
- $\delta(t - r/\beta)$: Οδευών παλμός που απομακρύνεται με ταχύτητα β.

$(x_1, x_2, x_3) \to (r, \theta, \varphi)$

$x_1 = r\sin\varphi\cos\theta$

$x_2 = r\sin\varphi\sin\theta$

$x_3 = r\cos\phi$

$$\begin{aligned} \gamma_1 &= \sin\phi\cos\theta, \quad \theta\in[0,2\pi),\ \phi\in[0,\pi] \\ \gamma_2 &= \sin\phi\sin\theta \\ \gamma_3 &= \cos\phi \end{aligned}$$

$\gamma_i = \dfrac{x_i}{r}, \quad r>0$

$$g^S_{i3} = \frac{\delta_{i3}-\gamma_i\gamma_3}{4\pi\rho\beta^2}\cdot\frac{1}{r}\cdot\delta(t-r/\beta)$$

$$[-\gamma_1\gamma_3,\ -\gamma_2\gamma_3,\ 1-\gamma_3^2] = \underset{\sim}{c}$$

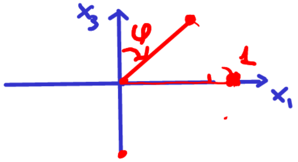
$$\|\underset{\sim}{c}\|_2^2 = \sin^2\phi\cos^2\theta\cos^2\phi + \sin^2\phi\sin^2\theta\cos^2\phi + (1-\cos^2\phi)^2$$

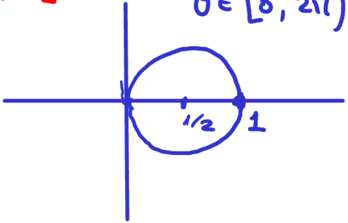
$$= \sin^2\phi\cos^2\phi(\cos^2\theta+\sin^2\theta) + \sin^4\phi =$$

$$= \sin^2\phi\cos^2\phi + \sin^4\phi = \sin^2\phi(\cos^2\phi+\sin^2\phi) = \sin^2\phi$$

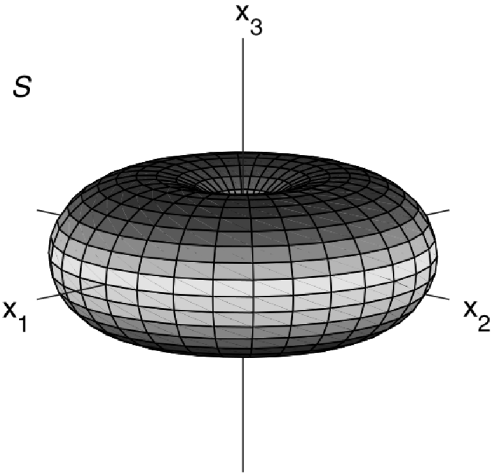
$\|\underset{\sim}{c}\|_2 = |\sin\phi|$ $\quad\theta = 0\quad$ $\phi \in [0,\pi]$

$\theta \in [0, 2\pi)$

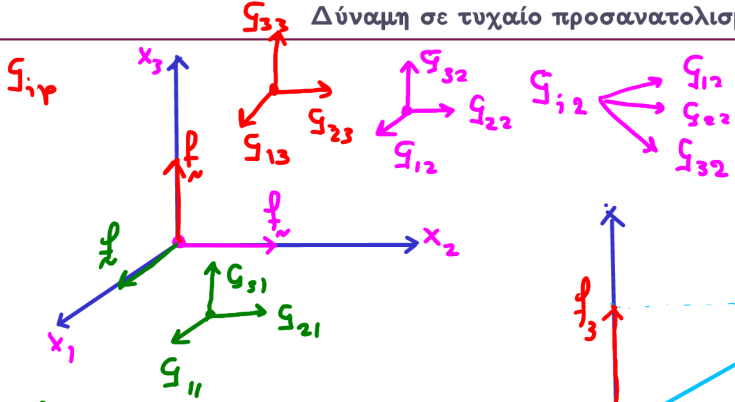




S-wave Πρότυπο ακτινοβολίας



Δύναμη σε τυχαίο προσανατολισμό

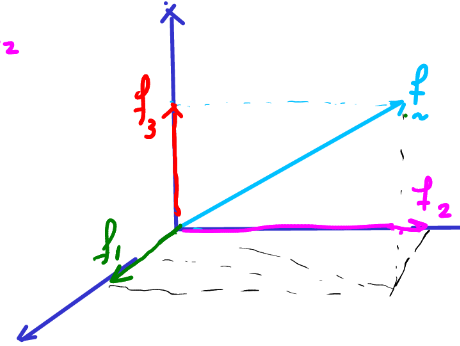


$$\underset{\sim}{f} = (f_1, f_2, f_3)$$

$$u_i = f_1 * G_{i1} \quad , \; i = 1,2,3$$

$$u_i = f_2 * G_{i2}$$

$$u_i = f_3 * G_{i3}$$



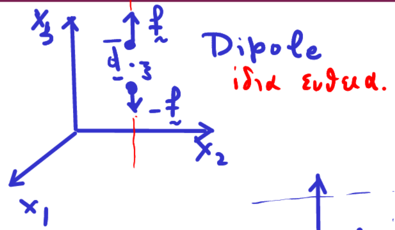
$$\boxed{u_i = \sum_{q=1}^{3} f_q * G_{iq}}$$

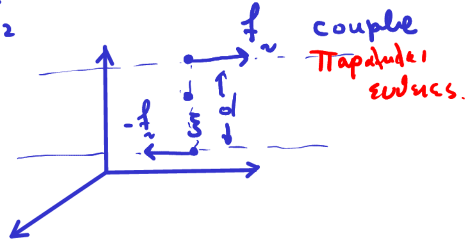
$$\Updownarrow$$

$$\boxed{u_i = f_q * G_{iq}}$$

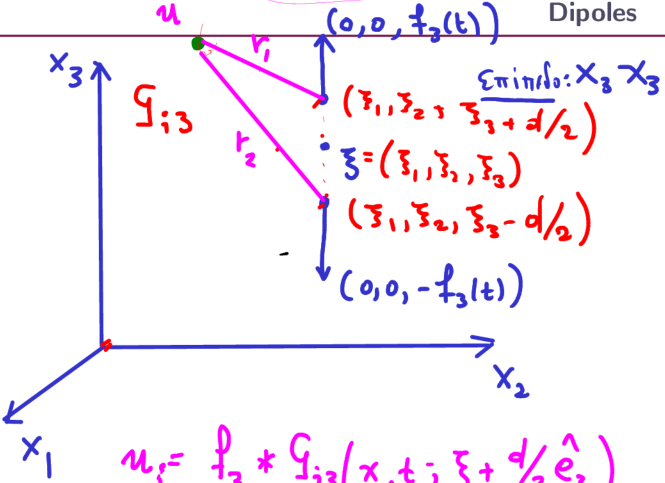
$$u_i = f_1 * G_{i1} + f_2 * G_{i2} + f_3 * G_{i3}$$

$$i = 1,2,3$$





- Ζεύγος παράλληλων Δυνάμεων.
- Σημεία εφαρμογής σε μικρή απόσταση.
- Αντίθετη φορά



$(x_1 = 0)$

$(0,0,f_3)$: $G_{i3} * f_3 \quad G_{i3}(\underset{\sim}{x}, t; \underset{\sim}{\xi} + d/2\, \hat{e}_3)$

$(0,0,-f_3)$: $-G_{i3} * f_3 \quad G_{i3}(\underset{\sim}{x}, t; \underset{\sim}{\xi} - d/2\, \hat{e}_3)$

$$u_i = f_3 * G_{i3}(\underset{\sim}{x}, t; \underset{\sim}{\xi} + d/2\, \hat{e}_3) - f_3 * G(\underset{\sim}{x}, t; \underset{\sim}{\xi} - d/2\, \hat{e}_3) = f_3\, d\, \frac{G_{i3}(\underset{\sim}{x}, t; \underset{\sim}{\xi} + d/2\, \hat{e}_3) - G_{i3}(\underset{\sim}{x}, t; \underset{\sim}{\xi} - d/2\, \hat{e}_3)}{d}$$

$d \to 0$, $f_3 \to \infty$ s.t. $d \cdot f_3 = M_{33} < \infty \longrightarrow = M_{33} \frac{\partial}{\partial \xi_3} G_{i3}(\underset{\sim}{x}, t; \underset{\sim}{\xi})$

Άσκηση 2 φυλ.

$$u_{xx} - c^{-2} u_{tt} + f(x) = 0 \qquad x>0$$

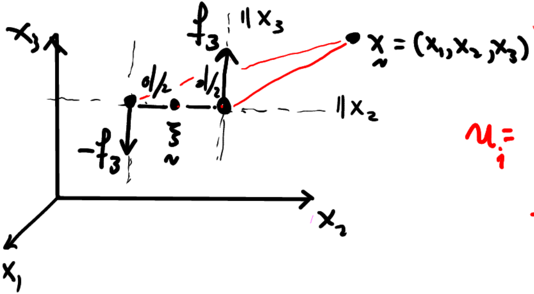
$$u(x,0) = g(x), \quad u_t(x,0) = h(x)$$

$$+ \; u(0,t) = 0$$

$$\rightarrow \quad x \in \mathbb{R}. \qquad \tilde{f} = \begin{cases} f, & x>0 \\ 0, & x=0 \\ -f, & x<0 \end{cases}$$

$$\tilde{g} = \begin{cases} g, & x>0 \\ 0, & x=0 \\ -g, & x<0 \end{cases} \qquad h = \begin{cases} h \\ 0 \\ -h \end{cases}$$

$$u = \tilde{u}\big|_{x>0} \qquad\qquad \tilde{u} \quad x \in \mathbb{R}$$

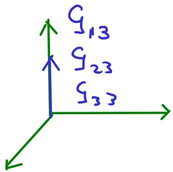


$$u_i = f_3 * G_{i3}(\underset{\sim}{x}, t; \underset{\sim}{\xi} + d/2\, \hat{e}_2)$$
$$\quad - f_3 * G_{i3}(\underset{\sim}{x}, t; \underset{\sim}{\xi} - d/2\, \hat{e}_2)$$

$$u_i(\underset{\sim}{x},t) = d f_3 * \frac{G_{i3}(\underset{\sim}{x},t;\underset{\sim}{\xi}+d/2\,\hat{e}_2) - G_{i3}(\underset{\sim}{x},t;\underset{\sim}{\xi}-d/2\,\hat{e}_2)}{d}$$

$$\left.\begin{array}{l} f_3 \to \infty \\ d \to 0 \end{array}\right\} \; d f_3 = M_{32} < \infty$$

$G_{ip}(\underset{\sim}{x},t;\underset{\sim}{\xi},\tau)$



$$u_i(\underset{\sim}{x},t) = M_{32} * \frac{\partial G_{i3}(\underset{\sim}{x},t;\underset{\sim}{\xi})}{\partial \xi_2}$$

Γενικά : $\underset{\approx}{M} = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix}$

$$u_i = M_{pq} * \frac{\partial G_{ip}}{\partial \xi_q} = \sum_{p=1}^{3} \sum_{q=1}^{3} M_{pq} * \frac{\partial G_{ip}}{\partial \xi_q}$$

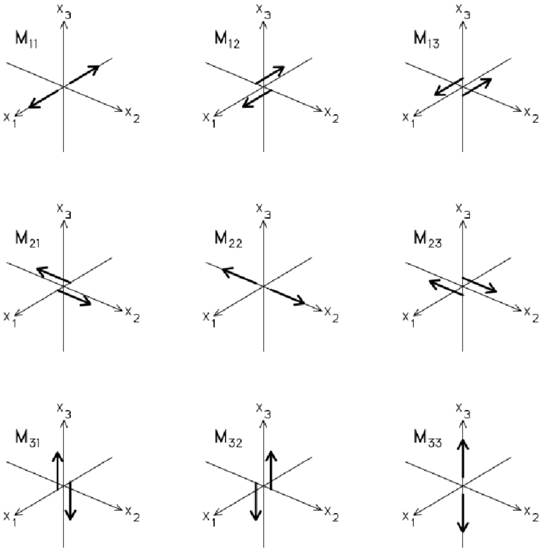
MEM-284: Κυματική Διάδοση

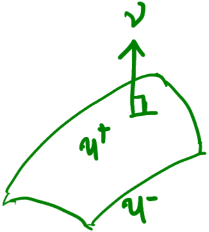
Τμήμα Μαθηματικών και Εφαρμοσμένων Μαθηματικών, Πανεπιστήμιο Κρήτης

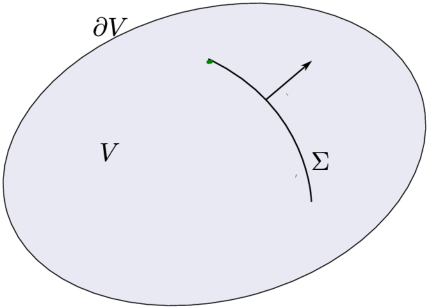
Κώστας Σμαραγδάκης (https://kesmarag.gitlab.io)

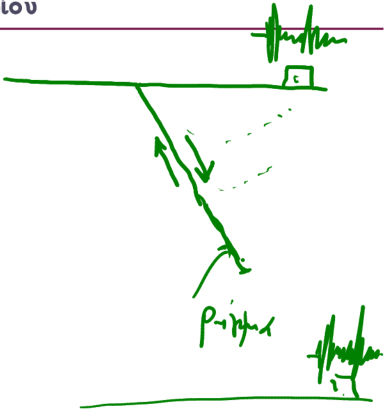
8η Διάλεξη - 18.3.2022

$\underset{\sim}{\xi} = \underset{\sim}{0}$





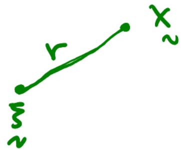




- Ασυνέχεια της μετατόπισης (εξάθρωση - dislocation) στην επιφάνεια Σ
- Την ασυνέχεια την συμβολίζουμε ως

$$[\mathbf{u}] = \mathbf{u}^{+} - \mathbf{u}^{-}$$

$$f_i = \delta_{ip}\,\delta(x - \xi)\,\delta(t)$$



Δύναμη πεδίου: $f_i = \delta_{ip}\delta(\mathbf{x} - \boldsymbol{\xi})\delta(t - \tau)$
Συνάρτηση Green: $\mathcal{G}_{ip}(\mathbf{x}, t; \boldsymbol{\xi}, \tau)$

- Η αρχή του χρόνου τ θα προσαρμόσει τους οδεύοντες παλμούς.

P-wave : $\delta(t - r/\alpha - \tau)$
S-wave : $\delta(t - r/\beta - \tau)$
Απόσταση σημείων $\mathbf{x}, \boldsymbol{\xi}$: $r = \|\mathbf{x} - \boldsymbol{\xi}\|_2$

$$\gamma_i = \frac{x_i - \xi_i}{r}$$

διεύθυνση της δύναμης

σημείο μέτρησης

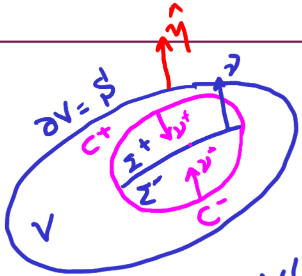
σημείο πηγής

χρόνος ενεργοποίησης

$$\mathcal{G}_{ip}(x, t; \xi, \tau)$$

i-συνιστώσα της μετατόπισης

χρόνος μέτρησης



$$\int_V \nabla \cdot \underset{\sim}{u}\, dV = \int_S \underset{\sim}{u} \cdot \hat{n}\, dS$$ χωρίς το Σ

$$[\underset{\sim}{u}] = \underset{\sim}{u}^+ - \underset{\sim}{u}^- \neq \underset{\sim}{0}$$

$$V' = V - C$$

$$\int_{V'} \nabla \cdot \underset{\sim}{u}\, dV' = \int_{S'} \underset{\sim}{u} \cdot \hat{n}\, dS' + \int_{C^+} \underset{\sim}{u} \cdot \hat{\nu}^+ dC^+ + \int_{C^-} \underset{\sim}{u} \cdot \hat{\nu}^- dC^-$$

$C^- \to \Sigma^-$

$C^+ \to \Sigma^+$

$V' \to V$

$\hat{\nu}^+ \to -\hat{\nu}$

$\hat{\nu}^- \to \hat{\nu}$

$\underset{\sim}{u}^+$

$\underset{\sim}{u}^-$

$$\int_V \nabla\cdot \underset{\sim}{u}\, dV = \int_{S'} \underset{\sim}{u}\cdot\hat{n}\, dS' = \int_{\Sigma^+} \underset{\sim}{u}^+\cdot\hat{\nu}\, d\Sigma^+ + \int_{\Sigma^-} \underset{\sim}{u}^-\cdot\hat{\nu}\, d\Sigma^-$$

$$\boxed{\int_V \nabla\cdot \underset{\sim}{u}\, dV = \int_{S'} \underset{\sim}{u}\cdot\hat{n}\, dS' - \int_{\Sigma} [\underset{\sim}{u}]\,\hat{\nu}\, d\Sigma}$$

Μέχρι τώρα έχουμε υποθέσει ισοτροπικά μέσα όπου

$$\tau_{ij} = \lambda \epsilon_{kk} \delta_{ij} + 2\mu \epsilon_{ij}$$

$$u_{j,i} = \frac{\partial u_j}{\partial x_i}$$

$$\epsilon_{ij} = (u_{j,i} + u_{i,j})/2$$

Γενική περίπτωση

$3 \times 3 \times 3 \times 3 = 81$

$$\underset{\approx\approx}{C} \qquad c_{ijpq}$$

$$\tau_{ij} = c_{ijpq} \epsilon_{pq}$$

$$\tau_{ij} = \tau_{ji}, \quad \epsilon_{pq} = \epsilon_{qp}$$

- Πόσες διαφορετικά στοιχεία μπορεί να λάβει ο τετραδιάστατος τανυστής με τους συντελεστές c_{ijpq}?

$6 \times 6 = 36$ διαφορετικά στοιχεία.

$$c_{ijpq} = c_{jipq}, \qquad c_{ijpq} = c_{ijqp}$$

- Τα διαφορετικά στοιχεία περιορίζονται στα 6*6=36.

Νόμος του Hook

$$\tau_{ij} = \sum_{p,q} c_{ijpq} u_{p,q}$$

Ισοτροπικά μέσα

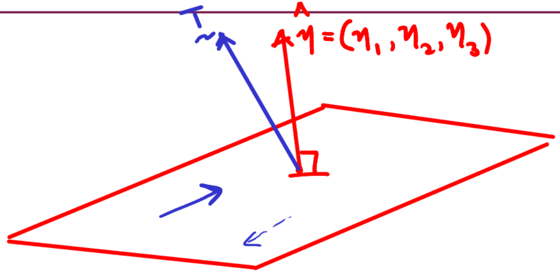
$$\delta_{ij} = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases}$$

$$c_{ijpq} = \lambda \delta_{ij}\delta_{pq} + \mu(\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp})$$

$$c_{1111} = \lambda \delta_{11}\delta_{11} + \mu(\delta_{11}\delta_{11} + \delta_{11}\delta_{11}) = \lambda + 2\mu$$

$$c_{1211} = \lambda \delta_{12}\delta_{11} + \mu(\delta_{11}\delta_{21} + \delta_{11}\delta_{21}) = 0$$

$C_{ijpq} \to \tau_{ij}$



- Κάθετο διάνυσμα $\mathbf{n}$
- Τανυστής τάσεων τ_{ij}

$$T_i = \tau_{ij} n_j$$

$$\underset{\sim}{T} = \begin{bmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

$$\begin{bmatrix} \tau_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \tau_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \tau_{33} \end{bmatrix}$$

<u>Παράδειγμα</u>

$$\underset{\approx}{\tau} = \begin{bmatrix} -p & 0 & 0 \\ 0 & -p & 0 \\ 0 & 0 & -p \end{bmatrix}$$

$\underset{\approx}{\tau}\,\hat{n} \leftarrow$ Πίεση.

MEM-284: Κυματική Διάδοση

Τμήμα Μαθηματικών και Εφαρμοσμένων Μαθηματικών, Πανεπιστήμιο Κρήτης

Κώστας Σμαραγδάκης (https://kesmarag.gitlab.io)

9η Διάλεξη - 24.3.2022

Νόμος του Hook

9 σταθερά

$$\tau_{ij} = \underline{c_{ijpq} u_{p,q}}$$

Ισοτροπικά μέσα

$$c_{ijpq} = \lambda \delta_{ij}\delta_{pq} + \mu(\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp})$$

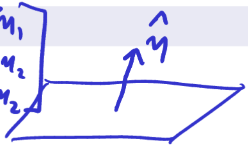
$$\delta_{ij} = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases}$$

δύναμη

Έλξη - Traction

- Κάθετο διάνυσμα $\mathbf{n}$
- Τανυστής τάσεων τ_{ij}

$$\begin{bmatrix} -p & & \\ & -p & \\ & & -p \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_2 \end{bmatrix}$$



$$T_i = \underline{\tau_{ij}} n_j$$

Ισοδύναμη δύναμη πεδίου

$$\tau_{ij,j} = \sum_{j=1}^{3} \tau_{ij,j} = \sum_{j=1}^{3} \frac{\partial}{\partial x_j} \tau_{ij}$$

$$\tau_{ij,j} + f_i = \rho \ddot{u}_i$$

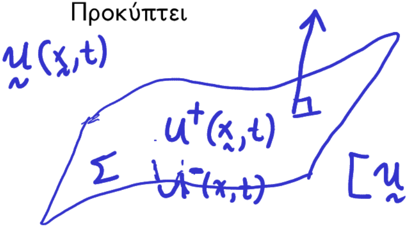
$$\hat{\nu} = \hat{\nu}(x,t)$$

$$\tau_{ij} = c_{ijpq} u_{p,q}$$

Προκύπτει

$$(c_{ijpq} u_{p,q})_{,j} + f_i = \rho \ddot{u}_i \quad \leftarrow \text{Κινητική Εξίσωση}$$

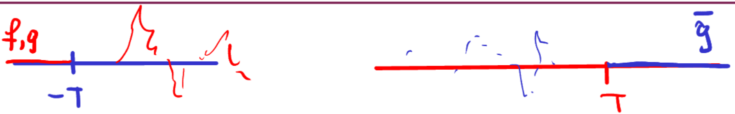
$$i = 1, 2, 3$$



$$[\underset{\sim}{u}(x,t)] = \underset{\sim}{u}^{+} - \underset{\sim}{u}^{-}$$

$t \in \mathbb{R}$

$T > 0$



Έστω διαφορετικές δυνάμεις f_i, g_i μηδενικές για $t < -T$

$$(c_{ijpq}u_{p,q})_{,j} + f_i = \rho \ddot{u}_i \leftarrow 1$$

$g \to v \qquad \bar{g} \to \bar{v}$

$$(c_{ijpq}v_{p,q})_{,j} + g_i = \rho \ddot{v}_i$$

Ορίζουμε

$$\bar{g}_i(\mathbf{x}, t) = g_i(\mathbf{x}, -t), \quad \bar{v}_i(\mathbf{x}, t) = v_i(\mathbf{x}, -t)$$

- $\bar{v}_i$ μηδενική για $t > T$

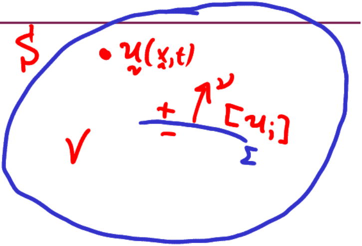
$$(c_{ijpq}\bar{v}_{p,q})_{,j} + \bar{g}_i = \rho \ddot{\bar{v}}_i \leftarrow 3$$

① $\underset{\sim}{f} \rightarrow \underset{\sim}{u}$ (μηδέν $t<-T$) εξίσωση u_i επί $\bar{v}_i$

$$(c_{ijpq}u_{p,q})_{,j}\bar{v}_i + f_i\bar{v}_i = \rho\ddot{u}_i\bar{v}_i$$

③ $\underset{\sim}{\bar{g}} \rightarrow \underset{\sim}{\bar{v}}$ (μηδέν $t>T$) εξίσωση $\bar{v}_i \cdot u_i$

$$(c_{ijpq}\bar{v}_{p,q})_{,j}u_i + \bar{g}_iu_i = \rho\ddot{\bar{v}}_iu_i$$



Σημείωση: έχω άθροισμα ως προς i

- Αφαιρούμε κατά μέλη και ολοκληρώνουμε χωρικά και χρονικά

$$\underbrace{\int_{\mathbb{R}} dt \int_V \{(c_{ijpq}\bar{v}_{p,q})_{,j}u_i - (c_{ijpq}u_{p,q})_{,j}\bar{v}_i\}dV}_{\mathrm{I}} + \underbrace{\int_{\mathbb{R}} dt \int_V \{\bar{g}_iu_i - f_i\bar{v}_i\}dV}_{\mathrm{II}} = \rho \underbrace{\int_{\mathbb{R}} dt \int_V \{\ddot{\bar{v}}_iu_i - \ddot{u}_i\bar{v}_i\}dV}_{\mathrm{III}}$$

III $\quad \dfrac{\partial}{\partial t}(\dot{\bar{v}}_i \cdot u_i) = \ddot{\bar{v}}_i u_i + \dot{\bar{v}}_i \dot{u}_i \qquad \dfrac{\partial}{\partial t}(\dot{u}_i\bar{v}_i) = \ddot{u}_i\bar{v}_i + \dot{\bar{v}}_i\dot{u}_i$

$$\frac{\partial}{\partial t}\left(\dot{\bar{v}}_i u_i - \dot{u}_i\bar{v}_i\right) = \ddot{\bar{v}}_i u_i - \ddot{u}_i\bar{v}_i$$

$$\mathrm{III} = \rho\int_V \int_{-\infty}^{+\infty} \frac{\partial}{\partial t}(\dot{\bar{v}}_i u_i - \dot{u}_i\bar{v}_i)\,dt\,dV = \rho\int_V \Big[\underset{\uparrow 0}{\overset{0\,\downarrow}{\dot{\bar{v}}_i u_i}} - \underset{\uparrow 0}{\overset{0\,\downarrow}{\dot{u}_i\bar{v}_i}}\Big]_{-\infty}^{+\infty} dV = 0$$

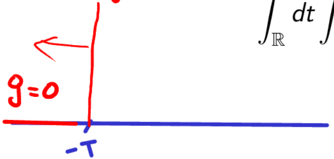
Ισοδύναμη δύναμη πεδίου

$$\frac{\partial}{\partial x_j}\left(u_i \bar{v}_{p,q}\right) = u_{i,j}\bar{v}_{p,q} + u_i \bar{v}_{p,qj}$$

$$\frac{\partial}{\partial x_j}\left(\bar{v}_i u_{p,q}\right) = \bar{v}_{i,j} u_{p,q} + \bar{v}_i u_{p,qj}$$

II

$$\int_{\mathbb{R}} dt \int_V \{c_{ijpq}(u_i\bar{v}_{p,q} - \bar{v}_i u_{p,q})\}_{,j} dV = \int_{\mathbb{R}} dt \int_V \{f_i\bar{v}_i - \bar{g}_i u_i\} dV$$



$$g_i(\mathbf{x},t) = \delta_{in}\delta(\mathbf{x}-\boldsymbol{\xi})\delta(t+T)$$

$$\bar{g}_i(\mathbf{x},t) = g_i(\mathbf{x},-t) = \delta_{in}\delta(\underset{\sim}{x}-\underset{\sim}{\xi})\underbrace{\delta(-t+T)}_{=\,\delta(t-T)}$$

$$v_i(\underset{\sim}{x},t) = G_{in}(\underset{\sim}{x},t;\underset{\sim}{\xi},-T)$$

$$\bar{v}_i(\mathbf{x},t) = G_{in}(\underset{\sim}{x},-t;\xi,-T) = G_{in}(x,t;\xi,T)$$

$$\text{II} = \int_{\mathbb{R}} dt \int_V f_i G_{in}(x,t;\xi,T)\,dV - \int_{\mathbb{R}} dt \int_V \delta_{in}\delta(\underset{\sim}{x}-\underset{\sim}{\xi})\delta(t-T)\,u_i(\underset{\sim}{x},t)\,dV_{\underset{\sim}{x}}$$

II b

1 εαν $i = n$
0 αλλιως

$I_b = \delta_{in} u_i(\xi, T) = u_n(\xi, T)$

$\delta_{1n} u_1 + \delta_{2n} u_2 + \delta_{3n} u_3$

$\int_{\mathbb{R}} dt \int g_{in} f_i \, dV - u_n(\xi, T)$

$$\int_{\mathbb{R}} dt \int_V \{c_{ijpq}(u_i \bar{v}_{p,q} - \bar{v}_i u_{p,q})\}_{,j} dV = \underbrace{\int_{\mathbb{R}} dt \int_V \{f_i \bar{v}_i - \bar{g}_i u_i\} dV}$$

$$g_i(\mathbf{x}, t) = \delta_{in} \delta(\mathbf{x} - \boldsymbol{\xi}) \delta(t + \tau)$$

$$\bar{g}_i(\mathbf{x}, t) = g_i(\mathbf{x}, -t) =$$

$$\bar{v}_i(\mathbf{x}, t) =$$

- Θεωρούμε ασυνέχεια $[u_i]$ σε μια επιφάνεια Σ του χώρου.

$$\int_{\mathbb{R}} dt \int_V \{C_{ijpq}(u_i G_{pn,q} - G_{in} u_{p,q})\}_{,j} \, dV =$$

$$= \int_{\mathbb{R}} dt \int_S \cancelto{0}{\{C_{ijpq}(u_i G_{pn,q} - G_{in} u_{p,q}\} n_j \, dS}$$

$$- \int_{\mathbb{R}} dt \int_\Sigma \{C_{ijpq}([u_i] G_{pn,q} - G_{in} [u_{p,q}]\} \nu_j \, d\Sigma$$

Άρα με την υπόθεση ότι $f_i = 0 \ \forall i$

$$u_n(\xi, t) = \int_{\mathbb{R}} dt \int_\Sigma \{C_{ijpq}([u_i] G_{pn,q} - G_{in} [u_{p,q}]\} \nu_j \, d\Sigma$$

$$\int_{\mathbb{R}} dt \int_{\Sigma} C_{ijpq} \underline{[u_{p,q}]\, \nu_j} \quad G_{in}\, d\Sigma$$

$\cdot\, \underset{\sim}{x}$

δραση $\quad T_p = 0$

αντιδραση

$\dot{\underset{\sim}{\xi}}$

$$u_n(\underset{\sim}{\xi}, T) = \int_{\mathbb{R}} dt \int_{\Sigma_{\underset{\sim}{x}}} C_{ijpq} \left([u_i](\underset{\sim}{x}, t)\, G_{pn,q}(\underset{\sim}{x}, t; \underset{\sim}{\xi}, T) \right) \nu_j\, d\Sigma_{\underset{\sim}{x}}$$

$$\delta_{in}\, \delta(\underset{\sim}{x} - \underset{\sim}{\xi})\, \delta(\underbrace{t - T}_{t'}) \longrightarrow G_{in}(\underset{\sim}{x}, t; \underset{\sim}{\xi}, T) = G_{in}(\underset{\sim}{x}, \underbrace{t - T}_{t'}; \underset{\sim}{\xi})$$

$$\delta_{ni}\, \delta(\underset{\sim}{\xi} - \underset{\sim}{x})\, \delta(T - t) \longrightarrow G_{ni}(\underset{\sim}{\xi}, T; \underset{\sim}{x}, t)$$

$$u_n(\underset{\sim}{\xi}, T) = \int_{\mathbb{R}} dt \int_{\Sigma_{\underset{\sim}{x}}} C_{ijpq} \left([u_i](\underset{\sim}{x}, t)\, G_{np,q}(\underset{\sim}{\xi}, T; \underset{\sim}{x}, t) \right) \nu_j\, d\Sigma_{\underset{\sim}{x}}$$

$\underset{\sim}{\xi} \leftrightarrow \underset{\sim}{x}$ $\quad T \leftrightarrow t$ $\quad$ αλλαγή συμβολισμού.

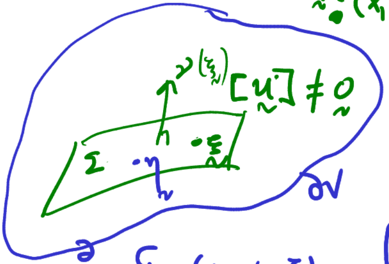
$$u_n(\underset{\sim}{x},t) = \int_{\mathbb{R}} dT \int_{\Sigma_{\underset{\sim}{\xi}}} C_{ijpq}\left([u_i](\underset{\sim}{\xi},T)\frac{\partial}{\partial \xi_q}\overbrace{G_{np}(\underset{\sim}{x},t;\underset{\sim}{\xi},T)}^{G_{np}(\underset{\sim}{x},t-T;\underset{\sim}{\xi})}\nu_j\, d\Sigma_{\underset{\sim}{\xi}}\right.$$

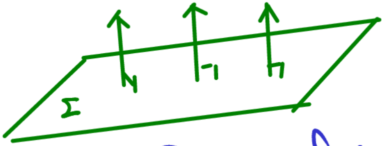
$$G_{np}(\underset{\sim}{x},t;\underset{\sim}{\xi},T) = G_{np}(\underset{\sim}{x},t-T;\underset{\sim}{\xi})$$

$$\boxed{u_n(\underset{\sim}{x},t) = \int_{\Sigma_{\underset{\sim}{\xi}}} \nu_j \underline{C_{ijpq}}\, [u_i](\underset{\sim}{\xi},t) * \frac{\partial}{\partial \xi_q} G_{np}(\underset{\sim}{x},t;\underset{\sim}{\xi})\, d\Sigma_{\underset{\sim}{\xi}}.}$$

Ισοδύναμη δύναμη πεδίου

$$\underset{\sim}{x} = (x_1, x_2, x_3) \longleftarrow u_n(\underset{\sim}{x}, t), \quad n = 1,2,3$$





$$\delta(\eta_1-\xi_1)\,\delta(\eta_2-\xi_2)\,\delta(\eta_3-\xi_3)$$

$$\frac{\partial}{\partial \xi_q} G_{np}(\underset{\sim}{x}, t; \underset{\sim}{\xi}) = \int_{V_\eta} \delta(\underset{\sim}{\eta} - \underset{\sim}{\xi}) \frac{\partial}{\partial \eta_q} G_{np}(\underset{\sim}{x}, t; \underset{\sim}{\eta})\, dV_\eta =$$

$$= \cancel{\delta(\underset{\sim}{\eta} - \underset{\sim}{\xi})\, G_{np}}^{\,0}(\underset{\sim}{x}, t; \eta)\Big|_{\partial V} - \int_{V_\eta} \frac{\partial}{\partial \eta_q} \delta(\underset{\sim}{\eta} - \underset{\sim}{\xi})\, G_{np}(\underset{\sim}{x}, t; \underset{\sim}{\eta})\, dV_\eta$$

$$u_n(\underset{\sim}{x},t) = -\int_{\Sigma_{\underset{\sim}{\xi}}} \nu_j C_{ijpq}[u_i](\underset{\sim}{\xi},t) * \int_{V_{\underset{\sim}{\eta}}} \frac{\partial}{\partial \eta_q} \delta(\underset{\sim}{\eta}-\underset{\sim}{\xi})\, G_{np}(\underset{\sim}{x},t;\underset{\sim}{\eta})\, dV_{\underset{\sim}{\eta}} \cdot d\Sigma_{\underset{\sim}{\xi}}$$

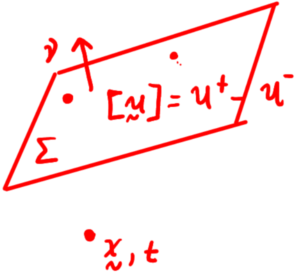
$$= \int_{V_{\underset{\sim}{\eta}}} G_{np}(\underset{\sim}{x},t;\underset{\sim}{\eta}) * \left[-\int_{\Sigma_{\underset{\sim}{\xi}}} \frac{\partial}{\partial \eta_q} \delta(\underset{\sim}{\eta}-\underset{\sim}{\xi})\, \nu_j C_{ijpq}[u_i](\underset{\sim}{\xi},t)\, d\Sigma_{\underset{\sim}{\xi}} \right] dV_{\underset{\sim}{\eta}}$$

$$\forall \underset{\sim}{\eta} \qquad \boxed{e_p = -\int_{\Sigma_{\underset{\sim}{\xi}}} \frac{\partial}{\partial \eta_q} \delta(\underset{\sim}{\eta}-\underset{\sim}{\xi})\, \nu_j C_{ijpq}[u_i](\underset{\sim}{\xi},t)\, d\Sigma_{\underset{\sim}{\xi}} \qquad p=1,2,3.}$$

MEM-284: Κυματική Διάδοση

Τμήμα Μαθηματικών και Εφαρμοσμένων Μαθηματικών, Πανεπιστήμιο Κρήτης

Κώστας Σμαραγδάκης (https://kesmarag.gitlab.io)



10η Διάλεξη - 7.4.2022

Θεώρημα αναπαράστασης

$$u_n(\mathbf{x}, t) = \int_{\Sigma_\xi} \nu_j c_{ijpq}[u_i](\boldsymbol{\xi}, t) * \frac{\partial}{\partial \xi_q} \mathcal{G}_{np}(\mathbf{x}, t; \boldsymbol{\xi}) d\Sigma_\xi$$

$$1 \cdot \mu D\, \delta(\xi_1 - \xi_{01})\, \delta(\xi_2 - \xi_{02}) * \left\{ \frac{\partial}{\partial \xi_1} G_{n3} + \frac{\partial}{\partial \xi_3} G_{n1} \right\} d\Sigma_\xi$$

$$= \mu D * \left\{ \frac{\partial}{\partial \xi_1} G_{n3}(\underset{\sim}{x}, t; \underset{\sim}{\xi_0}) + \frac{\partial}{\partial \xi_3} G_{n1}(\underset{\sim}{x}, t; \xi_0) \right.$$

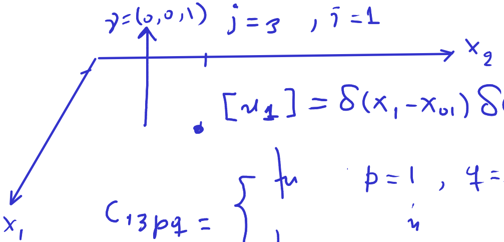
$$\underset{\sim}{\xi_0} = (\xi_{01}, \xi_{02}, 0)$$

Ισοδύναμη δύναμη

$$e_p(\boldsymbol{\eta}, t) = -\int_{\Sigma_\xi} \frac{\partial}{\partial \eta_q} \delta(\boldsymbol{\eta} - \boldsymbol{\xi}) \nu_j c_{ijpq}[u_i](\boldsymbol{\xi}, t) d\Sigma_\xi$$

Ομοιόμορφα και ισοτροπικά ελαστικά μέσα

$$c_{ijpq} = \lambda \delta_{ij} \delta_{pq} + \mu(\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp})$$

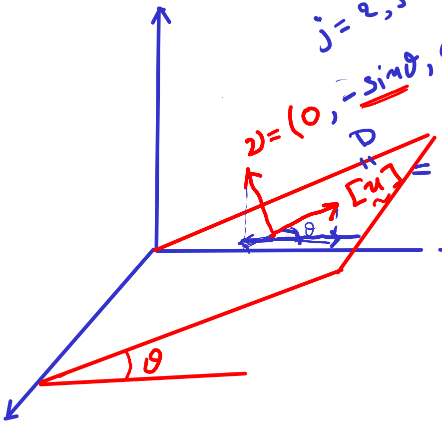


$j=3 \quad , \; i=1$

$$\mu(\delta_{1p}\delta_{3q}+\delta_{1q}\delta_{3p})$$

$$[u_1] = \delta(x_1-x_{01})\,\delta(x_2-x_{02}).D \quad \text{for} \quad D\neq 0.$$

$$C_{13pq} = \begin{cases} \mu & p=1\ ,\ q=3 \\ & \text{"} \\ \mu & p=3 \quad q=1. \end{cases}$$



$j = 2, 3$

$\nu = (0, -\sin\theta, \cos\theta)$

$i = 2, 3$

$[\underset{\sim}{u}] = (0, D\cos\theta, D\sin\theta)$

$$C_{ijpq} \to C_{22pq},\; C_{23pq},\; C_{32pq},\; C_{33pq}$$

$$C_{\underline{22}pq} = \lambda \delta_{22} \delta_{pq} + \mu(\delta_{2p}\delta_{2q} + \delta_{2q}\delta_{2p}) \to \begin{cases} \lambda & p = q \neq 2 \\ \lambda + 2\mu & p = q = 2 \end{cases}$$

$$C_{33pq} = \begin{cases} \lambda & p = q \neq 3 \\ \lambda + 2\mu & p = q = 3 \end{cases}$$

$$C_{23pq} = \begin{cases} \mu \,, & p=2\,,\ q=3 \\ \mu \,, & p=3\,,\ q=2 \end{cases} = C_{32pq}$$

$i=j=2$

$(p=q=2)$

$$u_n = \int_\Sigma (\lambda+2\mu)\cdot\underbrace{(-\sin\theta)}_{\nu_2}\underbrace{D\cos\theta}_{[u_2]} * \frac{\partial}{\partial \xi_2} G_{n2}\, d\Sigma_\xi \; +$$

$i=j=3$

$(p=q=3)$

$$+ \int_\Sigma (\lambda+2\mu)\cdot\underbrace{\cos\theta}_{\nu_3}\cdot\underbrace{D\sin\theta}_{[u_3]} * \frac{\partial}{\partial \xi_3} G_{n3}\, d\Sigma_\xi$$

$(i=2,\ j=3)$

$$+ \int_\Sigma \mu\cdot D\cos\theta\cos\theta * \left(\frac{\partial}{\partial \xi_2} G_{n3} + \frac{\partial}{\partial \xi_3} G_{n2}\right) d\Sigma_\xi$$

$(i=3,\ j=2)$

$$+ \int_\Sigma \mu\cdot(-\sin\theta)\sin\theta\, D * \left(\frac{\partial}{\partial \xi_2} G_{n3} + \frac{\partial}{\partial \xi_3} G_{n2}\right) d\Sigma_\xi$$

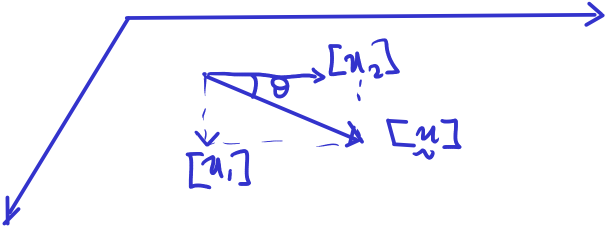
$$+\int_{\Sigma}^{i=j=2 \quad p=q=1} \to 0$$

$$+\int_{\Sigma}^{i=j=2 \quad p=q=3} \lambda \cdot (-\sin\theta)\, D\cos\theta * \frac{\partial}{\partial \xi_3} q_{n_3}$$

$$+\int^{i=j=3 \quad p=q=1} \to 0$$

$$+\int_{\Sigma}^{i=j=3 \quad p=q=2}$$

$[u_i]$ $\quad \underline{i = 2, 1} \quad$ $j = 3$



Για ομοιόμορφο και ισοτροπικό μέσο υπολογίστε τις σταθερές c_{13pq}

$$c_{13pq} = \lambda \cancelto{0}{\delta_{13}} \delta_{pq} + \mu \left(\delta_{1p}\delta_{3q} + \delta_{1q}\delta_{3p} \right)$$

$$= \begin{cases} \mu, & p=1, q=3 \text{ ή } p=3, q=1 \\ 0, & \text{διαφορετικά.} \end{cases}$$

Για ομοιόμορφο και ισοτροπικό μέσο υπολογίστε τις σταθερές c_{33pq}

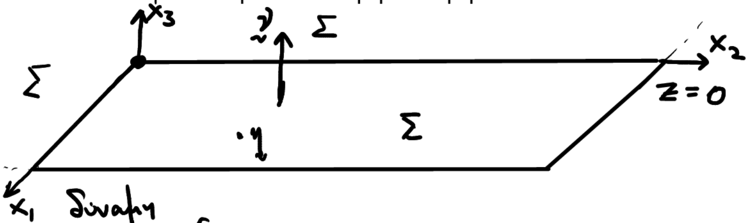
$$c_{33pq} = \lambda \overset{1}{\cancel{\delta_{33}}} \delta_{pq} + \mu \left(\delta_{3p}\delta_{3q} + \delta_{3q}\delta_{3p} \right) =$$

$$= \lambda \delta_{pq} + 2\mu \delta_{3p}\delta_{3q} = \begin{cases} \lambda, & p = q \neq 3 \\ \lambda + 2\mu, & p = q = 3 \\ 0, & \text{διαφορετικά.} \end{cases}$$

Θεωρήστε άπειρο ομοιόμορφο και ισοτροπικό ελαστικό μέσο. Για μια εξάρθρωση (dislocation) που συμβαίνει στο επίπεδο $z = 0$ που περιγράφεται ως

$$[u_1] = \delta(x_1)\delta(x_2)H(t),\ [u_2] = [u_3] = 0 \qquad z=0$$

Υπολογίστε την ισοδύναμη δύναμη.



Δύναμη

$$e_p(\eta,t) = -\int_{\Sigma_\xi} \frac{\partial}{\partial \eta_q}\delta(\eta - \xi)\,\nu_j\, c_{ijpq}\,[u_i](\xi,t)\,d\Sigma_\xi\ , \quad p \in \{1,2,3\}$$

$\nu = (0,0,1) \qquad \nu_j = \delta_{j3} \qquad [u_1] \neq 0 \qquad [u_2] = [u_3] = 0$

$$e_p(\underset{\sim}{\eta},t) = -\int_{\Sigma_{\underset{\sim}{\xi}}} \frac{\partial}{\partial \eta_q}\delta(\underset{\sim}{\eta}-\underset{\sim}{\xi})\,\nu_3\, \underline{C_{13pq}}\,[u_1](\underset{\sim}{\xi},t)\, d\Sigma_{\underset{\sim}{\xi}} =$$

$$C_{13pq} = \begin{cases} \mu, & p=1,\ q=3 \ \text{ή}\ \ p=3,\ q=1 \\ 0, & \text{διαφορετικά} \end{cases}$$

$$= -\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{\partial}{\partial \eta_q}\delta(\underset{\sim}{\eta}-\underset{\sim}{\xi})\, C_{13pq}\, \boxed{\delta(\xi_1)\delta(\xi_2)}\, H(t)\, d\Sigma_{\underset{\sim}{\xi}} =$$

$$= -\frac{\partial}{\partial \eta_q}\left\{\delta(\eta_1)\delta(\eta_2)\delta(\eta_3)\right\} C_{13pq}\, H(t) \Rightarrow$$

$$\underset{\sim}{e} = \left(-\frac{\partial}{\partial \eta_3}\left\{\delta(\eta_1)\delta(\eta_2)\delta(\eta_3)\right\} C_{1313}\, H(t),\ 0,\ -\frac{\partial}{\partial \eta_1}\left\{\delta(\eta_1)\delta(\eta_2)\delta(\eta_3)\right\} C_{1331}\, H(t)\right)$$

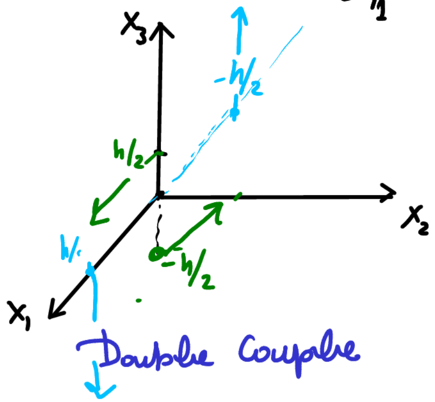
$$\underset{\sim}{e} = \left(-\mu H(t)\,\delta(\eta_1)\delta(\eta_2)\frac{\partial}{\partial \eta_3}\delta(\eta_3), 0, -\mu H(t)\,\delta(\eta_2)\delta(\eta_3)\frac{\partial}{\partial \eta_1}\delta(\eta_1) \right)$$

$\underline{p=1}$ Couple.

$$\frac{\partial}{\partial \eta_3}\delta(\eta_3) = \lim_{h\to 0} \frac{\delta(\eta_3 + h/2) - \delta(\eta_3 - h/2)}{h}$$

$\underline{p=3}$ Couple

$$\frac{\partial}{\partial \eta_1}\delta(\eta_1) = \lim_{h\to 0} \frac{\delta(\eta_1 + h/2) - \delta(\eta_1 - h/2)}{h}$$

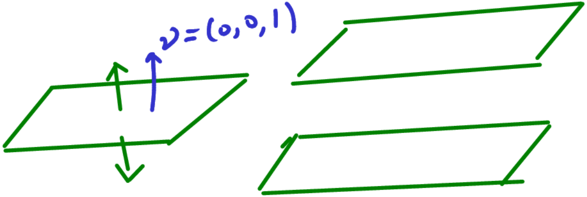


Double Couple

Θεωρήστε άπειρο ομοιόμορφο και ισοτροπικό ελαστικό μέσο. Για μια εξάρθρωση (dislocation) που συμβαίνει στο επίπεδο $z = 0$ που περιγράφεται ως

$$[u_1] = [u_2] = 0, \; [u_3] = \delta(x_1)\delta(x_2)H(t)$$

Υπολογίστε την ισοδύναμη δύναμη.



$i = 3 = j$

$$e_p(\underset{\sim}{\eta},t) = -\int_{\Sigma_\xi} \frac{\partial}{\partial \eta_q} \delta(\underset{\sim}{\eta} - \underset{\sim}{\xi})\, \nu_j\, c_{ijpq} [u_i](\underset{\sim}{\xi},t)\, d\Sigma_{\underset{\sim}{\xi}}$$

C_{33pq}

$$e_p(\underset{\sim}{\eta}, t) = -\frac{\partial}{\partial \eta_q}\{\delta(\eta_1)\delta(\eta_2)\delta(\eta_3)\} C_{33pq}[u_3].$$

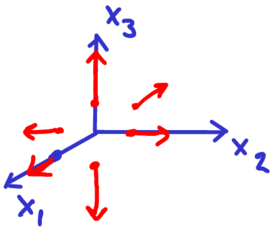
$$p = q \begin{cases} = 3 \\ \neq 3 \end{cases}$$

$p=1 \quad \frac{\partial}{\partial \eta_1}$ $\qquad$ $p=2 \quad \frac{\partial}{\partial \eta_2}$

$$\underset{\sim}{e} = \Big(-\lambda \frac{\partial}{\partial \eta_1}\{\delta(\eta_1)\}\delta(\eta_2)\delta(\eta_3)H(t), -\lambda\frac{\partial}{\partial \eta_2}\{\delta(\eta_2)\}\delta(\eta_1)\delta(\eta_3)H(t),$$

$$-(\lambda + 2\mu)\frac{\partial}{\partial \eta_3}\{\delta(\eta_3)\}\delta(\eta_1)\delta(\eta_2)H(t)\Big)$$

$p=3 \quad \frac{\partial}{\partial \eta_3}$



MEM-284: Κυματική Διάδοση

Τμήμα Μαθηματικών και Εφαρμοσμένων Μαθηματικών, Πανεπιστήμιο Κρήτης

Κώστας Σμαραγδάκης (https://kesmarag.gitlab.io)

11η Διάλεξη - 8.4.2022

$u_n = \int \mathcal{G} * f$

Θεώρημα αναπαράστασης

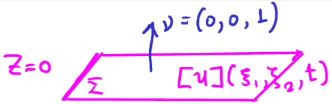
$$u_n(\mathbf{x}, t) = \int_{\Sigma_\xi} \nu_j c_{ijpq}[u_i](\boldsymbol{\xi}, t) * \frac{\partial}{\partial \xi_q} \mathcal{G}_{np}(\mathbf{x}, t; \boldsymbol{\xi}) d\Sigma_\xi$$

Ισοδύναμη δύναμη

$$\boxed{e_p(\boldsymbol{\eta}, t)} = - \int_{\Sigma_\xi} \frac{\partial}{\partial \eta_q} \delta(\boldsymbol{\eta} - \boldsymbol{\xi}) \nu_j c_{ijpq}[u_i](\boldsymbol{\xi}, t) d\Sigma_\xi \qquad \eta \in \Sigma$$

Ομοιόμορφα και ισοτροπικά ελαστικά μέσα

$$c_{ijpq} = \lambda \delta_{ij} \delta_{pq} + \mu(\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp})$$



$\bullet\ u(x,t)$

Μετατόπιση

$$g_{np}(x,t;\xi) * e_p(\xi,t) = \int_{\mathbb{R}} g_{np}(x,t-\tau;\xi)\, e_p(\xi,\tau)\, d\tau = g_{np}(x,t;\xi,\tau)$$

$$u_n(\mathbf{x},t) = \int_{V_0} \mathcal{G}_{np}(\mathbf{x},t;\boldsymbol{\xi}) * e_p(\boldsymbol{\xi},t)\, dV_{\xi}$$

- V_0 : όγκος στον οποίο η $\mathbf{e} = (e_1, e_2, e_3)$ είναι μη μηδενική.

Από την άσκηση 3 της διάλεξης 10 είχαμε

$z=0$ $\quad \iint_{-\infty}^{+\infty}$ $\quad \underbrace{d\xi_1\, d\xi_2}_{d\Sigma}$

$$\mathbf{e} = \left(-\mu H(t)\delta(\eta_1)\delta(\eta_2)\frac{\partial}{\partial \eta_3}\delta(\eta_3),\ 0,\ -\mu H(t)\delta(\eta_2)\delta(\eta_3)\frac{\partial}{\partial \eta_1}\delta(\eta_1)\right)$$

$$u_n(x,t) = \int_\Sigma \int_{\mathbb{R}} g_{np}(x,t;\xi,\tau)\, e_p(\xi,\tau)\, d\tau\, dV_\xi = -\mu \int_\Sigma \int_{\mathbb{R}} g_{n1}(x,t,\xi,\tau) H(\tau)\delta(\xi_1)\delta(\xi_2)\frac{\partial}{\partial \xi_3}\delta(\xi_3)\, d\tau\, d\Sigma_\xi$$

$$-\mu \int_\Sigma \int_{\mathbb{R}} g_{n3}(x,t,\xi,\tau) H(\tau)\delta(\xi_2)\delta(\xi_3)\frac{\partial}{\partial \xi_1}\delta(\xi_1)\, d\tau\, d\Sigma_\xi$$

$$= -\mu \frac{\partial \delta(\xi_3)}{\partial \xi_3} \int_{\Sigma} \int_0^{\infty} G_{n1}(\underset{\sim}{x}, t; \underset{\sim}{\xi}, \tau)\, \delta(\xi_1)\delta(\xi_2)\, d\tau\, d\Sigma_{\underset{\sim}{\xi}} \; - \underbrace{\mu\, \delta(\xi_3) \int_{\Sigma} \int_0^{\infty} G_{n3}\, \delta(\xi_2) \frac{\partial}{\partial \xi_1} \delta(\xi_1)\, d\tau\, d\Sigma_{\xi}}_{\circledast}$$

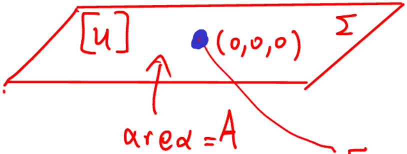
$$= -\mu \frac{\partial}{\partial \xi_3} \delta(\xi_3) \int_0^{\infty} G_{n1}(\underset{\sim}{x}, t; (0,0,\xi_3), \tau)\, d\tau \; + \circledast$$

$$\circledast = -\mu\, \delta(\xi_3) \int_0^{\infty} d\tau \int_{\Sigma} G_{n3}(\underset{\sim}{x}, t; \underset{\sim}{\xi}, \tau)\, \delta(\xi_2) \frac{\partial}{\partial \xi_1} \delta(\xi_1)\, d\Sigma_3 =$$

$$= \mu\, \delta(\xi_3) \int_0^{\infty} d\tau\, \frac{\partial}{\partial \xi_1} G_{n3}(\underset{\sim}{x}, t; (0,0,\xi_3), \tau)$$

Προσέγγιση μη σημειακής ασυνέχειας με σημειακή πηγή. μ

$$[u_1(\xi_1,\xi_2,t)],\ [u_2] = [u_3] = 0, \quad (\xi_1,\xi_2)\in\Sigma$$



$$\overline{[u]}(t) = \frac{1}{A}\int_\Sigma [u_1](\xi_1,\xi_2,t)\,d\xi_1 d\xi_2$$

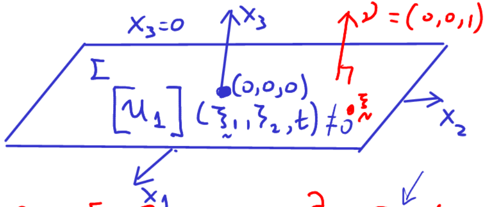
$$[u_1(\xi_1,\xi_2,t)] = \frac{1}{\mu}\underbrace{\mu\,\overline{[u]}(t)\,A}_{}\ \delta(\xi_1)\delta(\xi_2)$$

$M_0(t) \leftarrow$ ροπή της ισοδύναμης δύναμης

$$= \frac{1}{\mu}M_0(t)\,\delta(\xi_1)\delta(\xi_2)$$

$u_n(\underset{\sim}{x},t) = \,;$

example:



$\underset{\sim}{x}$

$[u_2]=[u_3]=0$

$$u_n(\underset{\sim}{x},t) = \int_{\Sigma_\xi} \nu_j\, C_{ijpq}\, [u_i](\underset{\sim}{\xi},t) * \frac{\partial}{\partial \xi_q} G_{np}(\underset{\sim}{x},t;\underset{\sim}{\xi})\, d\Sigma_{\underset{\sim}{\xi}}$$

$i=1 \quad j=3$

$p=1,\, q=3$

ή

$p=3,\, q=1.$

$C_{13pq} = \mu$, αν $p=1, q=3$ ή $p=3, q=1$.

$$u_n(\underset{\sim}{x},t) = \int_{\Sigma_\xi} \mu\, [u_1](\underset{\sim}{\xi},t) * \left\{ \frac{\partial}{\partial \xi_1} G_{n3}(\underset{\sim}{x},t;\underset{\sim}{\xi}) + \frac{\partial}{\partial \xi_3} G_{n1}(\underset{\sim}{x},t;\underset{\sim}{\xi}) \right\} d\Sigma_{\underset{\sim}{\xi}}$$

$G_{np} = \underline{G^P_{np}} + G^S_{np}$, $\quad G^P_{np} \xrightarrow{\theta.\alpha\nu\alpha\pi.} u^P_n$, $\quad G^S_{np} \longrightarrow u^S_n$

$u_n = u^P_n + u^S_n$

$f*(g_1+g_2) = f*g_1 + f*g_2 \qquad \int g_1+g_2 = \int g_1 + \int g_2$

Μελέτη για τα p-waves

$$g^{p}_{np} = \frac{\gamma_p}{4\pi\rho\alpha^2}\frac{1}{r}\gamma_n\,\delta(t - r/\alpha), \quad \gamma_i = \frac{x_i - \xi_i}{r}$$

$$r = \left((x_1-\xi_1)^2 + (x_2-\xi_2)^2 + (x_3-\xi_3)^2\right)^{1/2}$$

$$\frac{\partial \gamma_i}{\partial \xi_q} = \frac{\partial}{\partial \xi_q}\left(\frac{x_i - \xi_i}{r}\right) = \frac{\frac{\partial}{\partial \xi_q}(x_i - \xi_i)\, r - (x_i - \xi_i)\frac{\partial}{\partial \xi_q} r}{r^2} =$$

$$= \frac{-\delta_{iq}\, r - (x_i - \xi_i)\frac{\partial}{\partial \xi_q} r}{r^2} \quad (*)$$

$$\frac{\partial}{\partial \xi_q}\cancelto{0}{x_i} - \frac{\partial \xi_i}{\partial \xi_q} = \delta_{iq}$$

$$\frac{\partial r}{\partial \xi_q} = \frac{\partial}{\partial \xi_q}\left\{\left((x_1-\xi_1)^2 + (x_2-\xi_2)^2 + (x_3-\xi_3)^2\right)^{1/2}\right\} = -\frac{1}{\cancel{2}} r^{-1}\cdot\cancel{2}(x_q - \xi_q) = -\frac{x_q - \xi_q}{r} = -\gamma_q$$

$$(*) = -\frac{\delta_{iq}}{r} - \frac{1}{r}\frac{x_i - \xi_i}{r}\cdot(-\gamma_q) = \frac{-\delta_{iq} + \gamma_i\gamma_q}{r}$$

$$\frac{\partial}{\partial \xi_q}\delta(t - r/\alpha) = \delta'(t - r/\alpha)\cdot\frac{\partial}{\partial \xi_q}(t - r/\alpha) = -\frac{1}{\alpha}\delta'(t - r/\alpha)\frac{\partial}{\partial \xi_q} r = \frac{\gamma_q}{\alpha}\delta'(t - r/\alpha)$$

$$\frac{\partial G^p_{n1}}{\partial \xi_3} = \frac{\gamma_1}{4\pi\rho\alpha^2} \frac{\partial}{\partial \xi_3} \underbrace{\left\{ \frac{1}{r}\gamma_n \delta(t - r/\alpha) \right\}}$$

$r >> 1$

$$\frac{\partial}{\partial \xi_3}\left\{\frac{1}{r}\right\}\gamma_n \delta(t - r/\alpha) + \frac{1}{r}\frac{\partial \gamma_n}{\partial \xi_3}\delta(t - r/\alpha) + \frac{1}{r}\gamma_n \frac{\partial}{\partial \xi_3}\delta(t - r/\alpha)$$

$$= \frac{\gamma_3}{r^2}\gamma_n \delta(t - r/\alpha) + \frac{-\delta_{n3} + \gamma_n\gamma_3}{r^2}\delta(t - r/\alpha) + \frac{1}{r}\gamma_n \frac{\gamma_3}{\alpha}\delta'(t - r/\alpha)$$

$$\overset{r>>1}{\approx} \frac{1}{r\alpha}\gamma_n\gamma_3\delta'(t - r/\alpha).$$

$$\frac{\partial G^p_{n1}}{\partial \xi_3} \approx \frac{\gamma_n\gamma_1\gamma_3}{4\pi\rho r\alpha^3}\delta'(t - r/\alpha)$$

$p = 1,\ q = 3$

όμοια

$$\frac{\partial G^p_{n3}}{\partial \xi_1} \approx \frac{\gamma_n\gamma_1\gamma_3}{4\rho r\alpha^3}\delta'(t - r/\alpha)$$

$$\left\{\frac{\partial G^p_{n1}}{\partial \xi_3} + \frac{\partial G^p_{n3}}{\partial \xi_1}\right\} \approx \frac{\gamma_n\gamma_1\gamma_3}{2\pi\rho r\alpha^3}\delta'(t - r/\alpha)$$

$r >> 1$

$$u_n^f(\underset{\sim}{x}, t) = \mu A \int_{\Sigma_{\underset{\sim}{\xi}}} [\bar{u}_1](t)\, \underline{\delta(\xi_1)\delta(\xi_2)} * \frac{\gamma_n \gamma_1 \gamma_3}{2\pi\rho r \alpha^3} \delta'(t - r/\alpha)\, d\Sigma_{\underset{\sim}{\xi}} =$$

$$= \mu A \frac{\gamma_n \gamma_1 \gamma_3}{2\pi\rho r \alpha^3} [\bar{u}_1](t) * \delta'(t - r/\alpha)$$

$r = \|x\|$

$\gamma_i = \dfrac{x_i}{\|x\|}$

$$[\bar{u}_1](t) * \delta'(t - r/\alpha) = \int_{\mathbb{R}} [\bar{u}_1](\tau)\, \delta'(t - \tau - r/\alpha)\, d\tau =$$

$$f(t) * g(t) = \int_{\mathbb{R}} f(\tau)\, g(t-\tau)\, d\tau = \int_{\mathbb{R}} g(\tau)\, f(t-\tau)\, d\tau = g(t) * f(t)$$

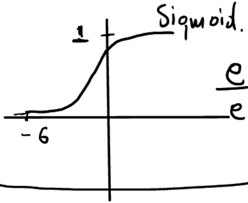
$$= \cancel{[\bar{u}_1](\tau)\, \delta(t - \tau - r/\alpha)\Big|_{-\infty}^{+\infty}}^{\,0} + \int_{\mathbb{R}} [\bar{u}_1]'(\tau)\, \delta(t - \tau - r/\alpha)\, d\tau$$

$$= + \int_{\mathbb{R}} [\overline{u_1}]'(\tau)\,\delta(t-\tau-r/\alpha)\,d\tau = + [\overline{u_1}]'(t - r/\alpha)$$

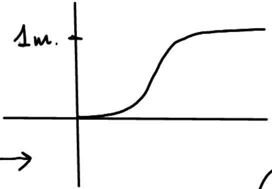
$$u^P_\eta(\underset{\sim}{x},t) = + \cancel{\mu} A \frac{\gamma_\eta \gamma_1 \gamma_3}{2\pi\rho r \alpha^3} [\overline{u_1}]'(t - r/\alpha), \quad \eta = 1,2,3$$

Παράδειγμα

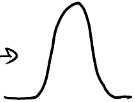
$$[\overline{u_1}](t) = \frac{e^{t-6}}{e^{t-6}+1}$$



$$\frac{e^t}{e^t+1}$$



$$[\overline{u_1}]'(t) = \frac{e^{t-6}(e^{t-6}+1) - e^{t-6}\, e^{t-6}}{(e^{t-6}+1)^2} = \frac{e^{t-6}}{(e^{t-6}+1)^2}$$



$$u_n^P(\underset{\sim}{x},t) = -\mu A \frac{\gamma_n \gamma_1 \gamma_3}{2\pi \rho r \alpha^3} \frac{e^{t-6-r/\alpha}}{\left(e^{t-6-r/\alpha}+1\right)^2}$$

$$\mu = \rho b^2$$

Παράδειγμα:

$$[\overline{u}](t) = H(t), \quad t \in \mathbb{R}$$

θ.δ.ο

$$H'(t) = \delta(t)$$

$$[\overline{u}]'(t) = \delta(t)$$

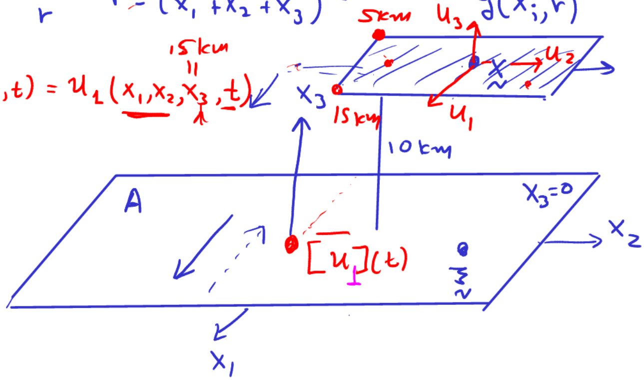
$$H(t) = \int_{-\infty}^{t} H'(\tau)\, d\tau = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

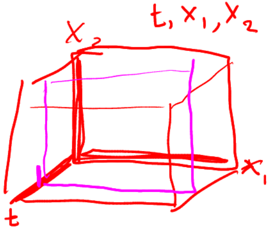
$$\int_{-\infty}^{t} \delta(\tau)\, d\tau = H(t)$$

$$u_n^P(\underset{\sim}{x},t) \overset{t}{\propto} \delta(t - r/\alpha)$$

$$\gamma_i = \frac{x_i}{r} \qquad r = \left(x_1^2 + x_2^2 + x_3^2\right)^{1/2}$$

$$u_1(\underset{\sim}{x}, t) = u_1(x_1, x_2, \overset{15\,km}{x_3}, t)$$



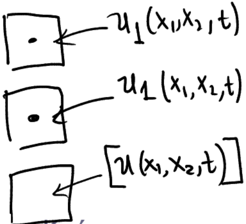


$u^p \in \mathbb{R}^{N\times N\times T}$

$u^s \in \mathbb{R}^{N\times N\times T}$

$u^p + u^s$

$n = 1$



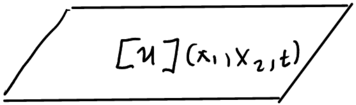
MEM-284: Κυματική Διάδοση

Τμήμα Μαθηματικών και Εφαρμοσμένων Μαθηματικών, Πανεπιστήμιο Κρήτης

Κώστας Σμαραγδάκης (https://kesmarag.gitlab.io)

12η Διάλεξη - 5.5.2022

$$\underset{\sim}{u}(\underset{\sim}{x},t) = \nabla \phi(\underset{\sim}{x},t) + \nabla \times \underset{\sim}{\psi}(\underset{\sim}{x},t)$$



Μετασχηματισμός Fourier

- Χρονικός $\quad t \longrightarrow \omega$

$$f(x,t) \xrightarrow{\mathcal{F}_{t\to\omega}} \hat{f}(x,\omega) = \int_{-\infty}^{+\infty} f(x,t)\, e^{i\omega t}\, dt$$

$$\hat{f}(x,\omega) \xrightarrow{\mathcal{F}^{-1}_{\omega\to t}} f(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{f}(x,\omega)\, e^{-i\omega t}\, d\omega$$

- Χωρικός $\quad x \longrightarrow k_x$

$$f(x,t) \xrightarrow{\mathcal{F}_{x\to k_x}} \hat{f}(k_x,t) = \int_{\mathbb{R}} f(x,t)\, e^{-ik_x x}\, dx$$

$$\hat{f}(k_x,t) \xrightarrow{\mathcal{F}^{-1}_{k_x\to x}} f(x,t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \hat{f}(k_x,t)\, e^{ik_x x}\, dk_x$$

$$\underset{\sim}{u}(\underset{\sim}{x},t) = \nabla\phi(\underset{\sim}{x},t) + \nabla\times\underset{\sim}{\psi}(\underset{\sim}{x},t)$$

$$\nabla^2\phi - \alpha^{-2}\ddot{\phi} = 0 \qquad \nabla^2\underset{\sim}{\psi} - \beta^{-2}\ddot{\underset{\sim}{\psi}} = \underset{\sim}{0}$$

$\underline{t \to w}$

$$\int_{\mathbb{R}} \ddot{\phi}(\underset{\sim}{x},t)\, e^{i\omega t}\, dt = (i\omega)^2 \int_{\mathbb{R}} \phi(\underset{\sim}{x},t)\, e^{i\omega t}\, dt = -\omega^2 \hat{\phi}(\underset{\sim}{x},\omega)$$

$$\int_{\mathbb{R}} \left(\nabla^2\phi(\underset{\sim}{x},t) - \alpha^{-2}\ddot{\phi}(x,t)\right) e^{i\omega t}\, dt = \nabla^2 \underbrace{\int_{\mathbb{R}} \phi(\underset{\sim}{x},t)\, e^{i\omega t}\, dt}_{\hat{\phi}(\underset{\sim}{x},\omega)} - \alpha^{-2} \cdot (-\omega^2)\hat{\phi}(x,\omega)$$

$$\hat{\phi}(\underset{\sim}{x},\omega) \doteq \phi(\underset{\sim}{x},\omega)$$

άρα $$\boxed{\nabla^2\phi(\underset{\sim}{x},\omega) + \alpha^{-2}\omega^2\phi(\underset{\sim}{x},\omega) = 0}$$

για κάθε ω λύνουμε

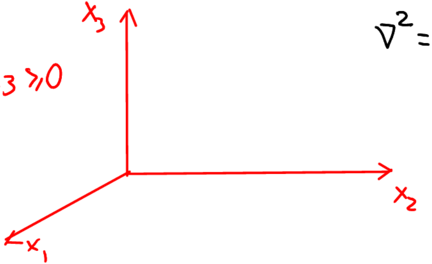
$$\boxed{\nabla^2 \phi + \left(\frac{\omega}{\alpha}\right)^2 \phi = 0} \quad \text{(Helmholtz)}$$

$$\nabla^2 \underset{\sim}{\psi} + \left(\frac{\omega}{\beta}\right)^2 \underset{\sim}{\psi} = \underset{\sim}{0} \longrightarrow \boxed{\nabla^2 \psi_i + \left(\frac{\omega}{\beta}\right)^2 \psi_i = 0}$$

$x_3 \geq 0$

Ψάχνουμε λύσεις της μορφής $\phi(\underset{\sim}{x}, \omega) = A \exp\{-ik_{x_1}x_1 - ik_{x_2}x_2 - i\nu x_3\}$

$x_3 \geq 0$



$$\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$$

$$\nabla^2 \phi = \underline{A}(-ik_{x_1})^2 \underline{\exp\{\dots\}} + A(-ik_{x_2})^2 \exp\{\dots\}$$

$$+ A(-i\nu)^2 \exp\{\dots\}$$

$$\nabla^2\phi = \left(-k_{x_1}^2 - k_{x_2}^2 - \nu^2\right)\phi$$

άρα $$\left(-k_{x_1}^2 - k_{x_2}^2 - \nu^2\right)\phi + \left(\frac{\omega}{\alpha}\right)^2\phi = 0 \quad , \quad k_\alpha = \frac{\omega}{\alpha}$$

$$\boxed{\nu^2 = \left(\frac{\omega}{\alpha}\right)^2 - k_{x_1}^2 - k_{x_2}^2} \Rightarrow \nu = \left(k_\alpha^2 - k_{x_1}^2 - k_{x_2}^2\right)^{1/2}$$

$$\nu \in \mathbb{C} \Rightarrow \nu = \nu_R + i\nu_I$$

$$\phi(x_1, x_2, x_3, \omega) = A\exp\left\{-ik_{x_1}x_1 - ik_{x_2}x_2 - i\overbrace{\left(k_\alpha^2 - k_{x_1}^2 - k_{x_2}^2\right)^{1/2}}x_3\right\}$$

θέλουμε.

$$\lim_{x_3\to\infty}\phi(x_1, x_2, x_3, \omega) = 0$$

$$\phi(\underset{\sim}{x}, \omega) \propto e^{-i\nu x_3} = e^{-i\nu_R x_3}\, e^{-i\cdot i\nu_I x_3} = e^{\nu_I x_3}\, e^{-i\nu_R x_3}$$

$$e^{\nu_I x_3} \xrightarrow[x_3\to\infty]{} 0 \quad \text{ανν} \quad \nu_I < 0$$

$x_3 < 0 \qquad \phi(\underset{\sim}{x}, \omega) = A' \exp\{-ik_{x_1}x_1 - ik_{x_2}x_2 + i\nu x_3\}$

$\nu = (k_\alpha^2 - k_{x_1}^2 - k_{x_2}^2)^{1/2}, \quad \nu_I < 0$

$\lim_{x_3 \to -\infty} \phi(x_1, x_2, x_3, \omega) = 0$

$x_3 \geq 0 \qquad -i\nu x_3$

$x_3 \leq 0 \qquad i\nu x_3$

Γ_{1d}

$x_3 \in \mathbb{R} \qquad \phi = A' \exp\{-ik_{x_1}x_1 - ik_{x_2}x_2 - i\nu|x_3|\}$

ohold

$\psi_i = B_i \exp\{-ik_{x_1}x_1 - ik_{x_2}x_2 - i\gamma|x_3|\}$

$\gamma = (k_\beta^2 - k_{x_1}^2 - k_{x_2}^2)^{1/2}, \quad \gamma_I < 0$

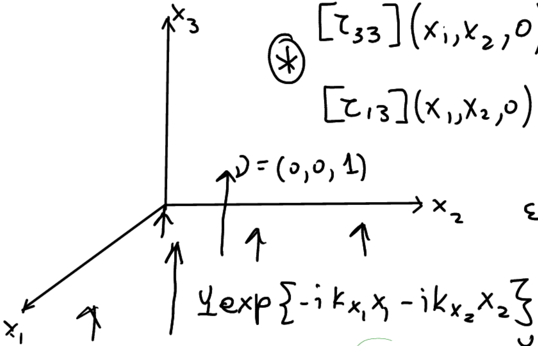
$$\underset{\sim}{u}(\underset{\sim}{x}, \omega) = \nabla \phi(\underset{\sim}{x}, \omega) + \nabla \times \underset{\sim}{\psi}(x, \omega)$$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \partial x_1 \phi \\ \partial x_2 \phi \\ \partial x_3 \phi \end{bmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial x_1 & \partial x_2 & \partial x_3 \\ \psi_1 & \psi_2 & \psi_3 \end{vmatrix}$$

$$\boxed{\begin{aligned} u_1 &= \underline{\partial x_1 \phi} + \partial x_2 \psi_3 - \partial x_3 \psi_2 \\ u_2 &= \underline{\partial x_2 \phi} - \partial x_1 \psi_3 + \partial x_3 \psi_1 \\ u_3 &= \underline{\partial x_3 \phi} + \partial x_1 \psi_2 - \partial x_2 \psi_1 \end{aligned}}$$

$$\phi = A \exp\{-i k_{x_1} x_1 - i k_{x_2} x_2 - i\nu |x_3|\}, \quad \mathrm{Im}\{\nu\} < 0$$

$$\underset{\sim}{\psi} = \underset{\sim}{B} \exp\{-i k_{x_1} x_1 - i k_{x_2} x_2 - i\gamma |x_3|\}, \quad \mathrm{Im}\{\gamma\} < 0$$



$$(*) \quad [\tau_{33}](x_1, x_2, 0) = \tau_{33}(x_1, x_2, 0^+) - \tau_{33}(x_1, x_2, 0^-) = -Y \exp\{-i k_{x_1} x_1 - i k_{x_2} x_2\}$$

$$[\tau_{13}](x_1, x_2, 0) = [\tau_{23}](x_1, x_2, 0) = 0$$

επίπεδο $x_3 = 0$

$\mu = \rho\beta^2$

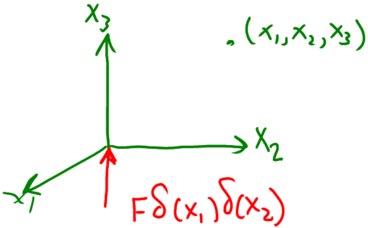
$$\mathrm{sgn}(x_3) = \begin{cases} 1, & x_3 > 0 \\ 0, & x_3 = 0 \\ -1, & x_3 < 0 \end{cases}$$

$$A = \mathrm{sgn}(x_3) \cdot \frac{Y}{2\mu k_\beta^2} \qquad k_\beta = \frac{\omega}{\beta}$$

$(*)$ + Νόμος Hooke $\rightarrow$ $\underset{\sim}{B}$: $B_1 = -k_{x_2} A/\gamma$, $\quad B_2 = k_{x_1} A/\gamma$, $\quad B_3 = 0$

$$[\tau_{33}](x_1,x_2,x_3=0) = \frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} dk_{x_1}\,dk_{x_2}\,[\tau_{33}](k_{x_1},k_{x_2},x_3=0)\exp\{-ik_{x_1}x_1 - ik_{x_2}x_2\}$$

$$[\tau_{33}](k_{x_1},k_{x_2},x_3=0) = \int_{\mathbb{R}}\int_{\mathbb{R}} [\tau_{33}](x_1',x_2',x_3=0)\exp\{ik_{x_1}x_1' + ik_{x_2}x_2'\}\,dx_1'\,dx_2'$$



Έστω

$$[\tau_{33}](x_1,x_2,x_3=0) = -\underline{F\delta(x_1)\delta(x_2)}$$

$$[\tau_{33}](k_{x_1},k_{x_2},0) = -F$$

$$[\tau_{33}](x_1,x_2,x_3=0) = -\frac{F}{4\pi^2}\int_{\mathbb{R}}\int_{\mathbb{R}} \exp\{-ik_{x_1}x_1 - ik_{x_2}x_2\}\,dk_{x_1}\,dk_{x_2}$$

$$= \int_{\mathbb{R}}\int_{\mathbb{R}} \left(-\frac{F}{4\pi^2}\right) \exp\left\{-ik_{x_1}x_1 - ik_{x_2}x_2\right\} dk_{x_1}\, dk_{x_2}$$

$\nu = (0,0,1) \leftarrow$ κάθετο διάνυσμα στο $x_3 = 0$

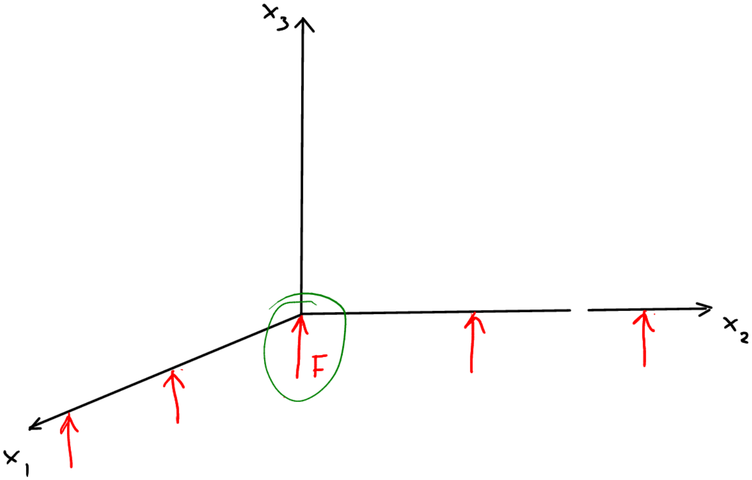
$$\phi^{(3)}(x_1,x_2,x_3=0,\omega) = \int_{\mathbb{R}}\int_{\mathbb{R}} \overbrace{\text{sgn}(x_3)\frac{F}{4\pi^2 \cdot 2\mu k_{\beta}^2}}^{A} \exp\left\{-ik_{x_1}x_1 - ik_{x_2}x_2 - i\nu|x_3|\right\} dk_{x_1}\, dk_{x_2}$$

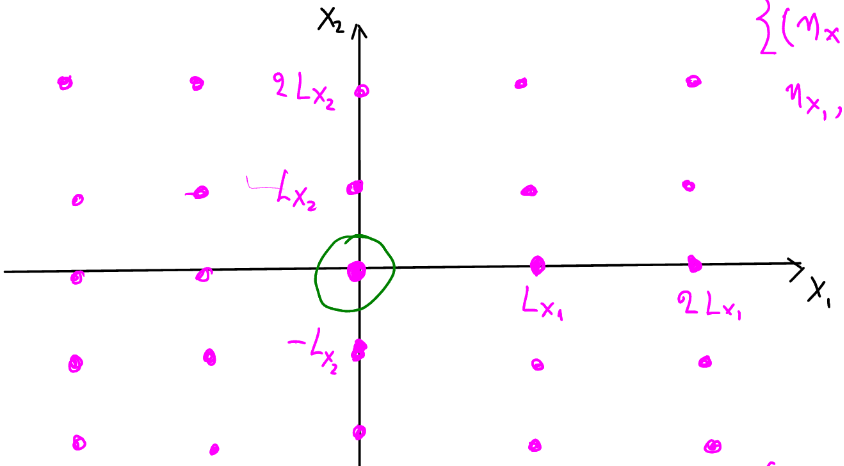
$$\psi_1^{(3)}(x_1,x_2,x_3=0,\omega) = -\int_{\mathbb{R}}\int_{\mathbb{R}} k_{x_2} A/\gamma \exp\left\{ \cdots -i\gamma|x_3|\right\} dk_{x_1}\, dk_{x_2}$$

Ομοίως $\psi_2^{(3)}, \psi_3^{(3)} = 0$

Θέλουμε $\iint \longrightarrow \sum\sum$

Discrete Wavenumber Representation Method





$$\{(n_{x_1}L_{x_1},\, n_{x_2}L_{x_2})\}$$

$$n_{x_1}, n_{x_2} \in \mathbb{Z}$$

Η λύση που αντιστοιχει σε δύναμη στο σημείο $(n_{x_1}L_{x_1},\, n_{x_2}L_{x_2})$

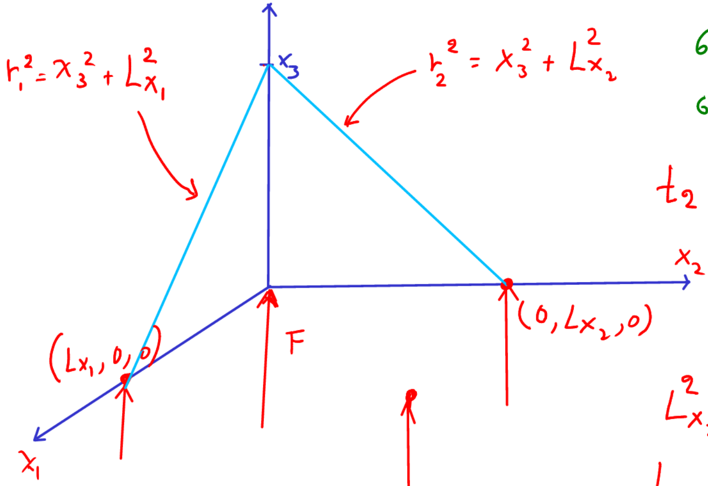
$$\phi^{x_3} \propto \int_{\mathbb{R}}\int_{\mathbb{R}} \exp\{-ik_{x_1}(x_1 - n_{x_1}L_{x_1}) - ik_{x_2}(x_2 - n_{x_2}L_{x_2}) - i\nu|x_3|\}\, dk_{x_1}dk_{x_2}$$

$$\sum_{n_{x_1}=-\infty}^{+\infty} \exp\{i k_{x_1} n_{x_1} L_{x_1}\} = 2\pi \sum_{n_{x_1}=-\infty}^{\infty} \delta(k_{x_1} L_{x_1} - 2\pi n_{x_1}) \rightarrow k_{x_1} = \frac{2\pi}{L_{x_1}} n_{x_1}$$

$$\phi^{x_3} \propto \sum_{n_{x_1}=-\infty}^{\infty} \sum_{n_{x_2}=-\infty}^{\infty} \int_{\mathbb{R}} \int_{\mathbb{R}} \exp\{-i k_{x_1} x_1 - i k_{x_2} x_2 - i\nu |x_3|\} \cdot \exp\{i k_{x_1} n_{x_1} L_{x_1}\} \cdot \exp\{i k_{x_2} n_{x_2} L_{x_2}\} \cdot dk_{x_1}\, dk_{x_2}$$

$$\phi^{x_3} \propto \sum_{n_{x_1}=-\infty}^{\infty} \sum_{n_{x_2}=-\infty}^{+\infty} \exp\left\{-i \frac{2\pi}{L_{x_1}} n_{x_1} x_1 - i \frac{2\pi}{L_{x_2}} n_{x_2} x_2 - i\nu |x_3|\right\}$$

$$\phi^{x_3} \propto \sum_{n_{x_1}=-N_1}^{N_1} \sum_{n_{x_2}=-N_2}^{N_2} \exp\left\{-i \frac{2\pi}{L_{x_1}} n_{x_1} x_1 - i \frac{2\pi}{L_{x_2}} n_{x_2} x_2 - i\nu |x_3|\right\}$$



Θέλουμε την επίδραση στις μετατοπίσεις της $F\delta(x_1)\delta(x_2)$ στους χρόνους $[0,T]$, $T>0$

$$t_2 = \frac{r_2}{\alpha} \geq T \Rightarrow \frac{r_2^2}{\alpha^2} \geq T^2 \Rightarrow$$

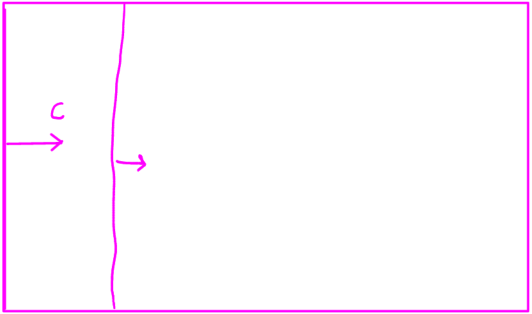
$$\Rightarrow \frac{x_3^2}{\alpha^2} + \frac{L_{x_2}^2}{\alpha^2} \geq T^2$$

$$L_{x_2}^2 \geq \alpha^2 T^2 - x_3^2$$

ή

$$L_{x_2} \geq \sqrt{\alpha^2 T^2 - x_3^2}$$

$$L_{x_1} \geq \sqrt{\alpha^2 T^2 - x_3^2}$$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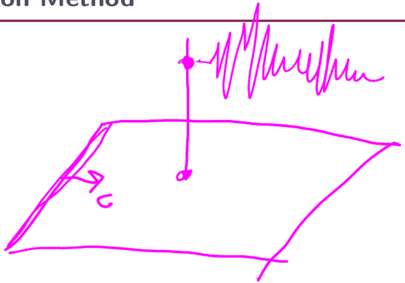
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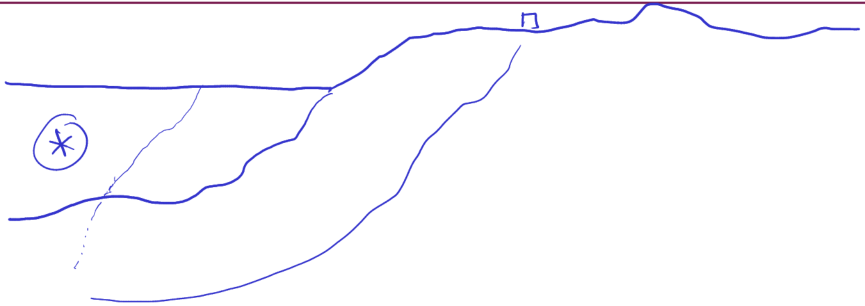
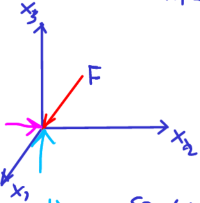


Discrete Wavenumber Representation Method

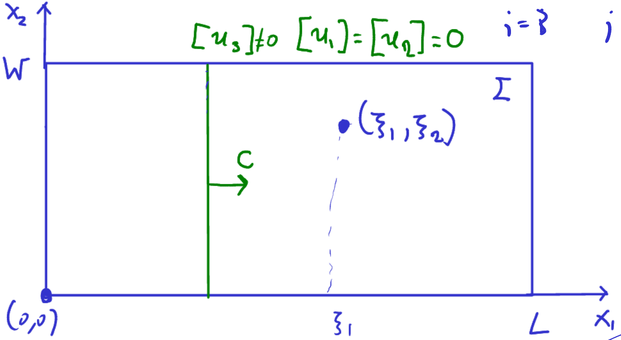
$$\phi^{x_1} = \frac{F}{2 L_{x_1} L_{x_2} \mu k_\beta^2} \sum_{n_{x_1}} \sum_{n_{x_2}} \frac{k_{x_1}}{\nu} \exp\left\{-i\frac{2\pi}{L_{x_1}} n_{x_1} x_1 - i \frac{2\pi}{L_{x_2}} n_{x_2} x_2 - i\nu|x_3|\right\}$$



$$\phi^{x_2} = \frac{F}{2 L_{x_1} L_{x_2} \mu k_\beta^2} \sum_{n_{x_1}} \sum_{n_{x_2}} \frac{k_{x_2}}{\nu} \exp\left\{ \quad \cdot \quad \right\}$$

$$\phi^{x_3} = \frac{\mathrm{sgn}(x_3)\, F}{2 L_{x_1} L_{x_2} \mu k_\beta^2} \sum_{n_{x_1}} \sum_{n_{x_2}} \exp\left\{ \quad \cdot \quad \right\}$$

$$\phi^{x_j} \propto \sum_{n_{x_1}} \sum_{n_{x_2}} \exp\left\{-i\frac{2\pi}{L_{x_1}} n_{x_1}(x_1-\xi_1) - i \frac{2\pi}{L_{x_2}} n_{x_2}(x_2-\xi_2) - i\nu|x_3-\xi_3|\right\}$$



$i = ?$ $\quad j = 3$

$$[u_3](\xi_1, \xi_2, t) = \begin{cases} \delta(t - \tau) & \text{στο } \Sigma \\ 0, & \text{αλλιώς} \end{cases}$$

τ - χρόνος αφίξεως του rupture front

$$\tau = \frac{\xi_1}{c}$$

$$\phi(x_1, x_2, x_3, \omega) = \frac{\mu}{F} \int_0^L \int_0^W \left(\frac{\partial \phi^{x_3}}{\partial \xi_1} + \frac{\partial \phi^{x_1}}{\partial \xi_3} \right) e^{-i\omega \frac{\xi_1}{c}} \, d\xi_2 \, d\xi_1$$

$$[u_3](\xi_1, \xi_2, t) \xrightarrow{\mathcal{F}_{t \to \omega}} \int_{\mathbb{R}} \delta\left(t - \frac{\xi_1}{c}\right) e^{-i\omega t} \, dt = e^{-i\omega \frac{\xi_1}{c}} = [u_3](\xi_1, \xi_2, \omega)$$

G_{qp} ϕ^{x_p} ← διευθ. της δυναμης $\underset{\sim}{u}^{p} = \nabla\phi$

$[u(\underset{\sim}{\xi}, t)]$ $[u(\xi, \omega)]$ $[u_i](\underset{\sim}{\xi}, t)$ $[u_i]$

$$\phi = \frac{1}{F}\int_{\Sigma_\xi} C_{ijpq}\, \nu_j\, [u_i](\underset{\sim}{\xi}, \omega)\, \frac{\partial \phi^{x_p}}{\partial \xi_q}\, d\Sigma_\xi = \phi(\underset{\sim}{x}, \omega)$$

F^2

$$\underset{\sim}{u} = \nabla\phi = \begin{bmatrix} \partial_{x_1}\phi \\ \partial_{x_2}\phi \\ \partial_{x_3}\phi \end{bmatrix}$$

$j = 1$ $i = 3$

$$\tilde{u}_1^P(x_1, x_2, x_3, \omega) = \partial x_1 \phi \qquad u = \nabla\phi + \nabla\times\psi$$

$$\tilde{u}_2^P(x_1, x_2, x_3, \omega) = \partial x_2 \phi \qquad \underbrace{\nabla\phi}_{u^P} \quad \underbrace{\nabla\times\tilde{\psi}}_{u^S}$$

$$\tilde{u}_3^P(x_1, x_2, x_3, \omega) = \partial x_3 \phi$$

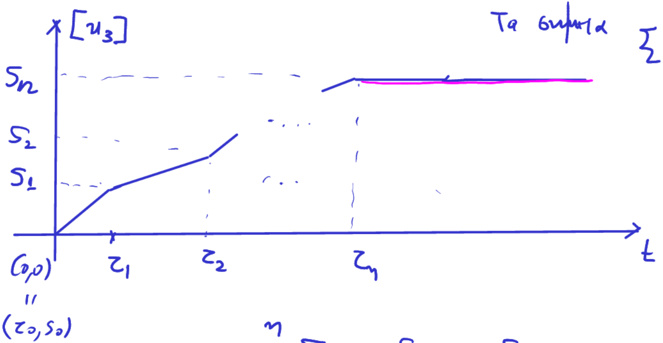
$$\tilde{u}_1^P = \frac{\operatorname{sgn}(x_3)}{2L_{x_1} L_{x_2} k_\beta^2} \sum_{n_{x_1}} \sum_{n_{x_2}} 2 k_{x_1}^2 e^{-i\nu|x_3|} \frac{\exp\left(i k_{x_1} L - i\frac{\omega}{c} L\right) - 1}{\omega/c - k_{x_1}} \cdot$$

$$\cdot \frac{\exp(i k_{x_2} W) - 1}{k_{x_2}} \cdot \exp\left(-i k_{x_1} x_1 - i k_{x_2} x_2\right)$$

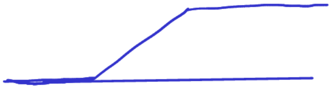
$$\tilde{u}_2^P = \frac{\operatorname{sgn}(x_3)}{2L_{x_1} L_{x_2} k_\beta^2} \sum\sum 2 k_{x_1} k_{x_2} \ldots\ldots$$

$$\tilde{u}_3^P = \frac{1}{2L_{x_1} L_{x_2} k_\beta^2} \sum\sum k_{x_1} 2\nu \ldots\ldots$$

Discrete Wavenumber Representation Method



Τα σημεία $\{(\tau_j, S_j)\}_{j=0,\dots,n}$ περιγράφουν την $[u_3]$



$$[u_3](\omega) = -\sum_{j=1}^{n} \left\{ \left[i\frac{S_j}{\omega} + \frac{S_j - S_{j-1}}{\omega^2(\tau_j - \tau_{j-1})} \right] \left(e^{-i\omega\tau_j} - e^{-i\omega\tau_{j-1}} \right) \right\} + \frac{i S_n e^{-i\omega\tau_n}}{\omega}$$

$$u_1^p = [u_3](\omega)\, \tilde{u}_1^p$$ ← συνέλιξη στο πεδίο των συχνοτήτων.

$$u_2^p = [u_3](\omega)\, \tilde{u}_2^p$$

$$u_3^p = [u_3](\omega)\, \tilde{u}_3^p$$

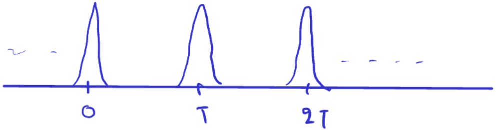
Διακριτό Μετασχηματισμό Fourier.

$t \in \{t_1, t_2, \dots, t_n\}$

$\omega \in \{\omega_1, \omega_2, \dots, \omega_n\}$

$$\delta_T(t) = \sum_{n=-\infty}^{+\infty} \delta(t - nT)$$

impulse train.



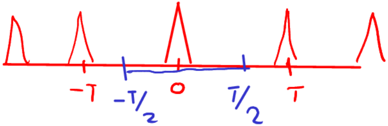
$$\xrightarrow{\mathcal{F}} \omega_s \sum_{k=-\infty}^{+\infty} \delta(\omega - k\omega_s)$$

όπου $\omega_s = \dfrac{2\pi}{T}$

$e^{i\frac{2k\pi}{T}t}$

$$\delta_T(t) = \sum_{k=-\infty}^{+\infty} c_k e^{ik\omega_s t} \Rightarrow c_k = \frac{1}{T}\int_{-T/2}^{T/2} \delta_T(t)\, e^{-ik\omega_s t}\, dt$$

$$c_k = \frac{1}{T}\int_{-T/2}^{T/2} \sum_{n=-\infty}^{+\infty} \delta(t-nT)\, e^{-ik\omega_s t}\, dt =$$



$$= \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)\, e^{-ik\omega_s t}\, dt = \frac{1}{T}$$

$$\delta_T(t) = \frac{1}{T}\sum_{k=-\infty}^{\infty} e^{ik\omega_s t}$$

Θέλουμε ν.δ.ο

για ένα $k \in \mathbb{Z}$

$$e^{ik\omega_s t} \xrightarrow{\mathcal{F}_{t\to\omega}} 2\pi\,\delta(\omega - k\omega_s)$$

$$\frac{1}{2\pi}\int 2\pi\,\delta(\omega - k\omega_s)\,e^{i\omega t}\,d\omega = e^{ik\omega_s t}$$

$$\delta_T(t) \xrightarrow{\mathcal{F}_{t\to\omega}} \omega_s \sum_{k=-\infty}^{+\infty} \delta(\omega - k\omega_s).$$

MEM-284: Κυματική Διάδοση

Τμήμα Μαθηματικών και Εφαρμοσμένων Μαθηματικών, Πανεπιστήμιο Κρήτης

Κώστας Σμαραγδάκης (https://kesmarag.gitlab.io)

13η Διάλεξη - 19.5.2022

$u(\omega) \to u(t)$ $\quad$ $u(t)\ ,\ t \geq 0$

$u(\omega_i) \longrightarrow u(t_i)$

Ιδιότητες συνέλιξης

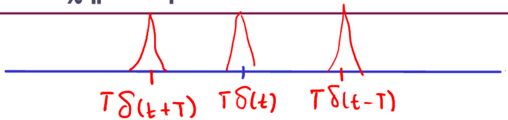
time-domain

$$f(t) * g(t) \overset{\mathcal{F}}{\leftrightarrow} f(\omega)g(\omega)$$

$$f(t)g(t) \overset{\mathcal{F}}{\leftrightarrow} \frac{1}{2\pi} f(\omega) * g(\omega)$$

frequency-domain

Έχουμε δείξει στο προηγούμενο μάθημα



$$\delta_T(t) = \sum_n \delta(t - nT) \leftrightarrow \delta_T(\omega) = \omega_s \sum_k \delta(\omega - k\omega_s), \quad \omega_s = 2\pi/T$$

Συνάρτηση δειγματοληψίας στο πεδίο του χρόνου

time-sampler

- Δοσμένο $T > 0 \rightarrow \omega_s$

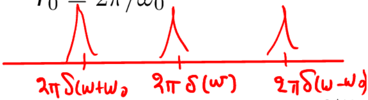
$$\underbrace{S_T(t)}_{\text{time}} = T\delta_T(t) = T\sum_m \delta(t - mT) \overset{\mathcal{F}}{\leftrightarrow} \underbrace{S_T(\omega)}_{\text{frequency}} = 2\pi \sum_k \delta(\omega - k\omega_s), \quad \omega_s = 2\pi/T$$

Συνάρτηση δειγματοληψίας στο πεδίο των συχνοτήτων

frequency-sampler

- Δοσμένο $\omega_0 > 0$

$$\underbrace{S_{\omega_0}(\omega)}_{\text{frequency}} = \omega_0 \sum_k \delta(\omega - k\omega_0) \overset{\mathcal{F}}{\leftrightarrow} \underbrace{S_{\omega_0}(t)}_{\text{time}} = \sum_m \delta(t - mT_0), \quad T_0 = 2\pi/\omega_0$$



$f(t),\ t\in\mathbb{R}$ $\qquad f_s(t)$

$$f_s(t) = f(t)\,T\sum_{m\in\mathbb{Z}}\delta(t-mT) = \sum_{m\in\mathbb{Z}} T\,\underbrace{f(t)\,\delta(t-mT)}^{\neq 0\quad t=mT} =$$

Έστω $f(t)$ με μετασχηματισμό Fourier $f(\omega)$.

$$= \sum_{m\in\mathbb{Z}} T\,f(mT)\,\delta(t-mT)$$

$$f_s(t) = f(t)S_T(t)$$

$$\left(f(t)\delta(t-mT) = \begin{cases} \infty\ f(t) & t=mT \\ 0 \end{cases} \right)$$

$$f_s(t) \leftrightarrow \tilde{f}(\omega) = \sum_{m\in\mathbb{Z}} Tf(mT)e^{-i\omega mT}$$

- Θα δείξουμε ότι $\tilde{f}(\omega)$ είναι περιοδική συνάρτηση με περίοδο ω_s.

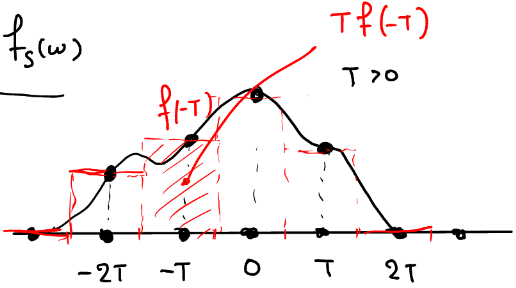
$$f_s(\omega) = \int_{\mathbb{R}}\sum_{m\in\mathbb{Z}} T\,f(mT)\,\delta(t-mT)\,e^{-i\omega t}\,dt = \sum_{m\in\mathbb{Z}} T\,f(mT)\int_{\mathbb{R}}\delta(t-mT)\,e^{-i\omega t}\,dt =$$

$$= \sum_{m\in\mathbb{Z}} T\,f(mT)\,e^{-i\omega mT}$$

$$\omega_s = \frac{2\pi}{T}$$

$$f_s(\omega + k\omega_s) = \underbrace{\sum_{m\in\mathbb{Z}} T f(mT)\, e^{-i\omega mT}}_{f_s(\omega)} e^{-ik\omega_s mT} = f_s(\omega)\, e^{-ikm2\pi} = f_s(\omega)$$

$$\tilde{f}(\omega) = f_s(\omega)$$



Διακριτό σήμα

$$f_d[mT] = Tf(mT)$$

Μετασχηματισμός Fourier διακριτού χρόνου

$$f_d[mT] \leftrightarrow \tilde{f}(\omega) = \sum_m f_d[mT]e^{-i\omega mT}$$

$$f_s(t) \xrightarrow{F} \tilde{f}(\omega) = \sum_m T f(mT) e^{-i\omega mT}$$

$$t = nT$$

$$f_d[nT] \rightarrow \sum_m f_d[mT]\, e^{-i\omega mT}$$

$$f(t) \leftrightarrow f(\omega) \qquad f*g = g*f$$

Θα δείξουμε ότι

$$\tilde{f}(\omega) = \sum_{\ell\in\mathbb{Z}} f(\omega - \ell\omega_s)$$

$$f_s(t) = f(t)\,S_T(t) \Rightarrow \tilde{f}(\omega) = \frac{1}{2\pi} f(\omega) * S_T(\omega) = \frac{1}{\cancel{2\pi}} f(\omega) * \cancel{2\pi}\sum_{\ell} \delta(\omega - \ell\omega_s)$$

$$S_T(\omega) = 2\pi \sum_{\ell\in\mathbb{Z}} \delta(\omega - \ell\omega_s)$$

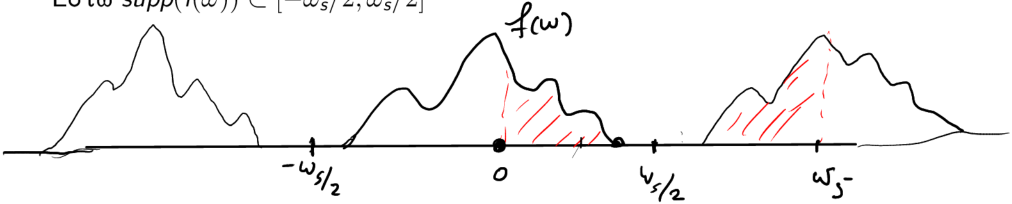
$$f_s(\omega)$$

$$\shortparallel$$

$$\tilde{f}(\omega) = \sum_{\ell\in\mathbb{Z}} f(\omega) * \delta(\omega - \ell\omega_s) = \sum_{\ell\in\mathbb{Z}} \int_{\mathbb{R}} f(\xi)\,\delta(\omega - \ell\omega_s - \xi)\,d\xi =$$

$$= \sum_{\ell\in\mathbb{Z}} f(\omega - \ell\omega_s)$$

Έστω $supp(f(\omega)) \subset [-\omega_s/2, \omega_s/2]$



$$T \to \omega_s$$

Μετασχηματισμός Fourier διακριτού χρόνου

$$\tilde{f}(\omega) = \sum_n f_d[nT] e^{-i\omega nT}$$

- Περιοδική με περίοδο ω_s

Αντίστροφος μετασχηματισμός Fourier διακριτού χρόνου

$$f_d[nT] = \frac{1}{\omega_s} \int_0^{\omega_s} \tilde{f}(\omega) e^{i\omega nT} d\omega$$

$$\operatorname{supp}(f(\omega)) \subset [-\omega_s/2, \omega_s/2]$$

$$f(t) = \frac{1}{2\pi} \int_{\mathbb{R}} f(\omega) e^{i\omega nT} d\omega = \frac{1}{2\pi} \int_{-\omega_s/2}^{\omega_s/2} f(\omega) e^{i\omega nT} d\omega =$$

$$= \frac{1}{2\pi}\int_{-\omega_s/2}^{\omega_s/2} \tilde{f}(\omega)\, e^{i\omega nT}\, d\omega = \frac{1}{2\pi}\int_0^{\omega_s} \tilde{f}(\omega)\, e^{i\omega nT}\, d\omega$$

$$T f(t) = \frac{T}{2\pi}\int_0^{\omega_s} \tilde{f}(\omega)\, e^{i\omega nT}\, d\omega \qquad t = nT$$

$$T f(nT) = \frac{1}{\omega_s}\int_0^{\omega_s} \tilde{f}(\omega)\, e^{i\omega nT}\, d\omega$$

$$\parallel$$

$$f_d[nT]$$

$W_0 \to T_0$

Mallat

A wavelet tour of signal processing.

Μετασχηματισμός Fourier διακριτης συχνότητας

$$f_d[k\omega_0] = \frac{1}{T_0}\int_0^{T_0} \tilde{f}(t) e^{-ik\omega_0 t} dt$$

Αντίστροφος μετασχηματισμός Fourier διακριτής συχνότητας

$$\tilde{f}(t) = \sum_k f_d[k\omega_0] e^{ik\omega_0 t}$$

- Περιοδική με περίοδο T_0

Στοιχεία Διακριτού Μετασχηματισμού Fourier

$$f(t) \xrightarrow{\mathcal{F}} f(\omega)$$

1) $T>0$ $\quad nT \leftarrow$ διακριτοί χρόνοι $\xrightarrow{\mathcal{F}} \tilde{f}(\omega)$ περίοδο $\omega_s = \frac{2\pi}{T}$

2) $\omega_0>0$ $\quad k\omega_0 \leftarrow$ διακριτές συχνότητες $\xrightarrow{\mathcal{F}^{-1}} \tilde{f}(t)$ περίοδο $T_0 = \frac{2\pi}{\omega_0}$

Χρησιμοποιούμε time-sampler $S_T(t)$ $-T \quad 0 \quad T \quad 2T$

$$f(t) \xrightarrow{\cdot S_T} f_d[nT] \xrightarrow{\mathcal{F}_{dt}} \tilde{f}(\omega) \xrightarrow{\cdot S_{\omega_0}} \tilde{f}_d[k\omega_0]$$

$\tilde{f}(\omega)$ περιοδική με περίοδο $\omega_s = \dfrac{2\pi}{T}$

εφαρμογή frequency sampler

Χρησιμοποιούμε frequency-sampler $-\omega_0 \quad 0 \quad \omega_0 \quad 2\omega_0$

$$f(\omega) \xrightarrow{\cdot S_{\omega_0}} f_d[k\omega_0] \xrightarrow{\mathcal{F}^{-1}_{d\omega}} \tilde{f}(t) \xrightarrow{\cdot S_T} \tilde{f}_d[nT]$$

$\tilde{f}(t)$ περιοδική με περίοδο $T_0 = \dfrac{2\pi}{\omega_0}$

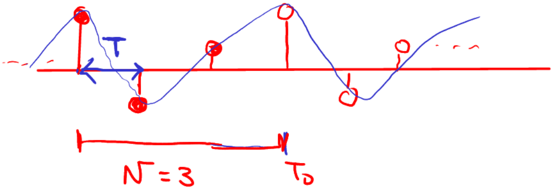
εφαρμογή time-sampler

⋮

$$\tilde{f}_d[nT] \underset{d\mathcal{F}^{-1}}{\overset{d\mathcal{F}}{\rightleftarrows}} \tilde{f}_d[k\omega_0]$$

$T = ; \rightarrow \omega_s = ;$

$T_0 = ; \rightarrow \omega_0 = ;$



$\exists N \in \mathbb{N}$ τ.ω $T_0 = NT$ $\exists K \in \mathbb{N}$ τ.ω $\omega_s = K\omega_0$

$$K = \frac{\omega_s}{\omega_0} = \frac{\frac{2\pi}{T}}{\frac{2\pi}{T_0}} = \frac{T_0}{T} = \frac{NT}{T} = N$$

Στοιχεία Διακριτού Μετασχηματισμού Fourier

Διακριτός μετασχηματισμός Fourier

$$\tilde{f}_d[k\omega_0] = \frac{1}{NT} \sum_{k=0}^{N-1} \tilde{f}_d[nT] e^{-ik\omega_0 nT}$$

Αντίστροφος διακριτός μετασχηματισμός Fourier

$$\tilde{f}_d[nT] = T \sum_{k=0}^{N-1} \tilde{f}_d[k\omega_0] e^{ik\omega_0 nT}$$

- $\hat{f}[k] = \tilde{f}_d[k\omega_0]$ $\hat{f}[k] = \tilde{f}_d[k\omega_0]$
- $f[n] = \tilde{f}_d[nT]$ $f[n] = \tilde{f}_d[nT]$

Διακριτός μετασχηματισμός Fourier

$$\hat{f}[k] = \frac{1}{NT} \sum_{n=0}^{N-1} f[n] e^{-ik(2\pi/N)}$$

Αντίστροφος διακριτός μετασχηματισμός Fourier

$$f[n] = T \sum_{k=0}^{N-1} \hat{f}[k] e^{ik(2\pi/N)}$$

Παράδειγμα

$0 \quad \omega_s \quad 2\omega_s$

Θέλουμε να υπολογίσουμε το πεδίο μετατοπίσεων για διάρκεια $10s$, στο ευρος συχνοτήτων $(0, 1)Hz$. Ποια θα ήταν μια καλή επιλογή για τα ω_0, T, N;

u_1, u_2, u_3

$\omega = 2\pi f$

radians Hz

$\omega \in [0, 2\pi]$

linspace$(0, 2\pi, 32)$

διαμέριση του $(0, 2\pi)$

$\omega_n \in \{ \ldots\ldots \}$

$\mathrm{supp}\, u(\omega) \subset [-\omega_s/2, \omega_s/2]$

$\boxed{2\pi \leq \omega_s/2}$

$\omega_s \geq 4\pi$

2π

$-\omega_s/2 \quad 0 \quad \omega_s/2$

$\omega_s = \frac{2\pi}{T}$

$\omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{10}$

$10 = T_0 = NT$

$T \leq 0.5$

$N \geq 20$

$N = 32$

$T = \frac{10}{32}$

$f_s = \frac{1}{T}$ sampling frequency.

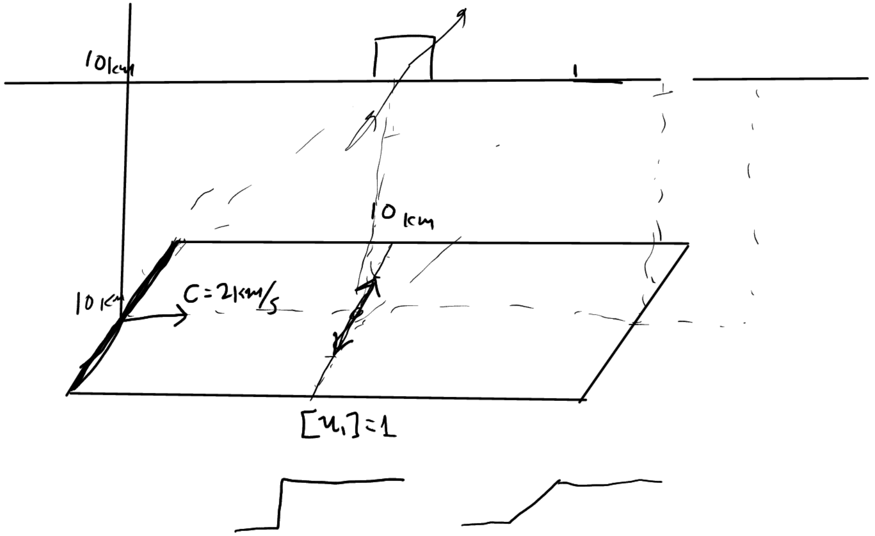
$\omega_0 = 0 \qquad \omega_1 = \frac{2\pi}{T} \qquad \omega_2 = 2\frac{2\pi}{T}$

10 s. $(0,1)\,Hz \longrightarrow (0, 2\pi)\ rad/s$

$\omega_s/2 \geq 2\pi \Rightarrow \omega_s \geq 4\pi$ $\omega_s = \frac{2\pi}{T} \geq 4\pi$

$T_0 = 10 = N \cdot T$ $T \leq 0.5\ s.$

$N \geq 20$



10km
10 km
10 km
C = 2km/s
[u₁] = 1