MEM-284: Κυματική Διάδοση

Τμήμα Μαθηματικών και Εφαρμοσμένων Μαθηματικών, Πανεπιστήμιο Κρήτης

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9η Διάλεξη - 24.3.2022

Νόμος του Hook

9 σταθερά

$$\tau_{ij} = \underline{c_{ijpq} u_{p,q}}$$

Ισοτροπικά μέσα

$$c_{ijpq} = \lambda \delta_{ij} \delta_{pq} + \mu (\delta_{ip} \delta_{jq} + \delta_{iq} \delta_{jp})$$

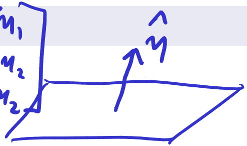
$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

Δύναμη

Έλξη - Traction

- Κάθετο διάνυσμα $\mathbf{n}$
- Τανυστής τάσεων τ_{ij}

$$\begin{bmatrix} -p & & \\ & -p & \\ & & -p \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$



$$T_i = \underline{\tau_{ij}} n_j$$

$$\tau_{ij,j} = \sum_{j=1}^{3} \tau_{ij,j} = \sum_{j=1}^{3} \frac{\partial}{\partial x_j} \tau_{ij}$$

$$\downarrow$$

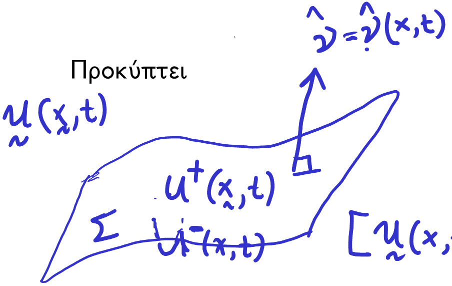
$$\tau_{ij,j} + f_i = \rho \ddot{u}_i$$

$$\tau_{ij} = c_{ijpq} u_{p,q}$$

Προκύπτει

$$(c_{ijpq} u_{p,q})_{,j} + f_i = \rho \ddot{u}_i \quad \leftarrow \text{Κυματική Εξίσωση}$$

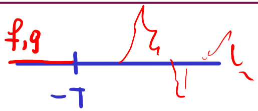
$$i = 1,2,3$$

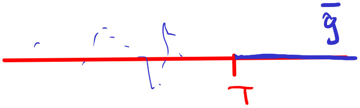


$$[\underset{\sim}{u}(x,t)] = \underset{\sim}{u}^+ - \underset{\sim}{u}^-$$

$t \in \mathbb{R}$

$T > 0$





Έστω διαφορετικές δυνάμεις f_i, g_i μηδενικές για $t < -T$

$$(c_{ijpq}u_{p,q})_{,j} + f_i = \rho \ddot{u}_i \leftarrow (1)$$

$g \to v \qquad \bar{g} \to \bar{v}$

$$(c_{ijpq}v_{p,q})_{,j} + g_i = \rho \ddot{v}_i$$

Ορίζουμε

$$\bar{g}_i(\mathbf{x}, t) = g_i(\mathbf{x}, -t), \quad \bar{v}_i(\mathbf{x}, t) = v_i(\mathbf{x}, -t)$$

- $\bar{v}_i$ μηδενική για $t > T$

$$(c_{ijpq}\bar{v}_{p,q})_{,j} + \bar{g}_i = \rho \ddot{\bar{v}}_i \leftarrow (3)$$

Ισοδύναμη δύναμη πεδίου

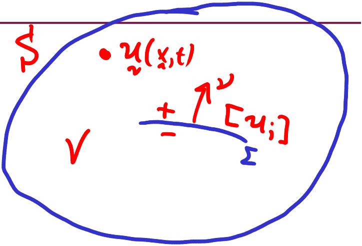
① $\underset{\sim}{f} \to \underset{\sim}{u}$ (μηδέν $t<-T$) εξίσωση u_i επί $\bar{v}_i$

$$(c_{ijpq}u_{p,q})_{,j}\bar{v}_i + f_i\bar{v}_i = \rho\ddot{u}_i\bar{v}_i$$

③ $\underset{\sim}{\bar{g}} \to \underset{\sim}{\bar{v}}$ (μηδέν $t>T$) εξίσωση $\bar{v}_i \cdot u_i$

$$(c_{ijpq}\bar{v}_{p,q})_{,j}u_i + \bar{g}_i u_i = \rho\ddot{\bar{v}}_i u_i$$

Σημείωση: έχω άθροισμα ως προς i



- Αφαιρούμε κατά μέλη και ολοκληρώνουμε χωρικά και χρονικά

$$\underbrace{\int_{\mathbb{R}} dt \int_V \{(c_{ijpq}\bar{v}_{p,q})_{,j}u_i - (c_{ijpq}u_{p,q})_{,j}\bar{v}_i\}dV}_{I} + \underbrace{\int_{\mathbb{R}} dt \int_V \{\bar{g}_i u_i - f_i\bar{v}_i\}dV}_{II} = \underbrace{\rho \int_{\mathbb{R}} dt \int_V \{\ddot{\bar{v}}_i u_i - \ddot{u}_i\bar{v}_i\}dV}_{III}$$

III $\quad \frac{\partial}{\partial t}(\dot{\bar{v}}_i \cdot u_i) = \ddot{\bar{v}}_i u_i + \dot{\bar{v}}_i \dot{u}_i \qquad \frac{\partial}{\partial t}(\dot{u}_i\bar{v}_i) = \ddot{u}_i\bar{v}_i + \dot{\bar{v}}_i\dot{u}_i$

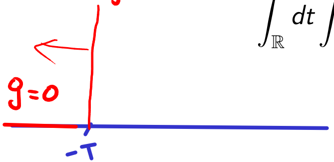
$$\frac{\partial}{\partial t}(\dot{\bar{v}}_i u_i - \dot{u}_i\bar{v}_i) = \ddot{\bar{v}}_i u_i - \ddot{u}_i\bar{v}_i$$

$$III = \rho\int_V \int_{-\infty}^{+\infty} \frac{\partial}{\partial t}(\dot{\bar{v}}_i u_i - \dot{u}_i\bar{v}_i)\,dt\,dV = \rho\int_V \Big[\overset{0}{\dot{\bar{v}}_i}\, \underset{0}{u_i} - \overset{0}{\dot{u}_i}\, \underset{0}{\bar{v}_i}\Big]_{-\infty}^{+\infty} dV = 0$$

$$\frac{\partial}{\partial x_j}\left(u_i \bar{v}_{p,q}\right) = u_{i,j}\bar{v}_{p,q} + u_i \bar{v}_{p,qj}$$

$$\frac{\partial}{\partial x_j}\left(\bar{v}_i u_{p,q}\right) = \bar{v}_{i,j} u_{p,q} + \bar{v}_i u_{p,qj}$$

$$\int_{\mathbb{R}} dt \int_V \{c_{ijpq}(u_i\bar{v}_{p,q} - \bar{v}_i u_{p,q})\}_{,j} dV = \overset{II}{\int_{\mathbb{R}} dt \int_V \{f_i\bar{v}_i - \bar{g}_i u_i\} dV}$$



$$g_i(\mathbf{x},t) = \delta_{in}\delta(\mathbf{x}-\boldsymbol{\xi})\delta(t+T)$$

$$\bar{g}_i(\mathbf{x},t) = g_i(\mathbf{x},-t) = \delta_{in}\delta(\underset{\sim}{x}-\underset{\sim}{\xi})\overbrace{\delta(-t+T)}^{\delta(t-T)}$$

$$v_i(\underset{\sim}{x},t) = G_{in}(\underset{\sim}{x},t;\underset{\sim}{\xi},-T)$$

$$\bar{v}_i(\mathbf{x},t) = G_{in}(\underset{\sim}{x},-t;\xi,-T) = G_{in}(x,t;\xi,T)$$

$$II = \int_{\mathbb{R}} dt \int_V f_i G_{in}(x,t;\xi,T)\, dV - \underbrace{\int_{\mathbb{R}} dt \int_V \delta_{in}\delta(\underset{\sim}{x}-\underset{\sim}{\xi})\delta(t-T)\, u_i(\underset{\sim}{x},t)\, dV_{\underset{\sim}{x}}}_{II_b}$$

1 εαν $i=n$
0 αλλιως

$$\text{II}_b = \delta_{in} u_i(\xi, T) = u_n(\xi, T)$$

$$\delta_{1n} u_1 + \delta_{2n} u_2 + \delta_{3n} u_3$$

$$\underbrace{\int_{\mathbb{R}} dt \int g_{in} f_i \, dV}\; - u_n(\xi, T)$$

$$\int_{\mathbb{R}} dt \int_V \{c_{ijpq}(u_i\bar{v}_{p,q} - \bar{v}_i u_{p,q})\}_{,j} dV = \int_{\mathbb{R}} dt \int_V \{f_i\bar{v}_i - \bar{g}_i u_i\} dV$$

$$g_i(\mathbf{x}, t) = \delta_{in}\delta(\mathbf{x} - \boldsymbol{\xi})\delta(t + \tau)$$

$$\bar{g}_i(\mathbf{x}, t) = g_i(\mathbf{x}, -t) =$$

$$\bar{v}_i(\mathbf{x}, t) =$$

- Θεωρούμε ασυνέχεια $[u_i]$ σε μια επιφάνεια Σ του χώρου.

$$\int_{\mathbb{R}} dt \int_V \{C_{ijpq}(u_i G_{pn,q} - G_{in} u_{p,q})\}_{,j}\, dV =$$

$$= \int_{\mathbb{R}} dt \int_S \cancelto{0}{\{C_{ijpq}(\dot{u}_i G_{pn,q} - G_{in} u_{p,q}\} n_j\, dS}$$

$$- \int_{\mathbb{R}} dt \int_\Sigma \{C_{ijpq}([u_i] G_{pn,q} - G_{in}[u_{p,q}]\}\nu_j\, d\Sigma$$

Άρα με την υπόθεση ότι $f_i = 0 \ \forall i$

$$u_n(\xi,t) = \int_{\mathbb{R}} dt \int_\Sigma \{C_{ijpq}([u_i] G_{pn,q} - G_{in}[\dot{u}_{p,q}]\}\nu_j\, d\Sigma$$

$$\int_{\mathbb{R}} dt \int_{\Sigma} C_{ijpq} \underline{[u_{p,q}] \nu_j} \; G_{in} \, d\Sigma$$

δράση $\quad T_p = 0$

αντίδραση

$\cdot \underset{\sim}{x}$

$\cdot \underset{\sim}{\xi}$

$$u_n(\underset{\sim}{\xi}, T) = \int_{\mathbb{R}} dt \int_{\Sigma_{\underset{\sim}{x}}} C_{ijpq} \left([u_i](\underset{\sim}{x}, t) \, G_{pn,q}(\underset{\sim}{x}, t; \underset{\sim}{\xi}, T) \right) \nu_j \, d\Sigma_{\underset{\sim}{x}}$$

$$\delta_{in} \, \delta(\underset{\sim}{x} - \underset{\sim}{\xi}) \, \delta(\underbrace{t - T}_{t'}) \longrightarrow G_{in}(\underset{\sim}{x}, t; \underset{\sim}{\xi}, T) = G_{in}(\underset{\sim}{x}, \underbrace{t - T}_{t'}; \underset{\sim}{\xi})$$

$$\delta_{ni} \, \delta(\underset{\sim}{\xi} - \underset{\sim}{x}) \, \delta(T - t) \longrightarrow G_{ni}(\underset{\sim}{\xi}, T; \underset{\sim}{x}, t)$$

$$u_n(\underset{\sim}{\xi}, T) = \int_{\mathbb{R}} dt \int_{\Sigma_{\underset{\sim}{x}}} C_{ijpq} \left([u_i](\underset{\sim}{x}, t) \, G_{np,q}(\underset{\sim}{\xi}, T; \underset{\sim}{x}, t) \right) \nu_j \, d\Sigma_{\underset{\sim}{x}}$$

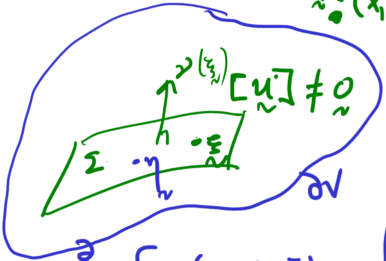
$\underset{\sim}{\xi} \leftrightarrow \underset{\sim}{x} \qquad T \leftrightarrow t$ αλλαγή συμβολισμού.

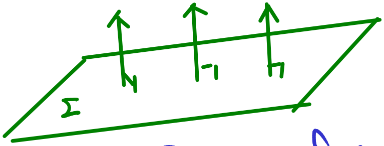
$$u_n(\underset{\sim}{x},t) = \int_{\mathbb{R}} dT \int_{\Sigma_{\underset{\sim}{\xi}}} C_{ijpq}\left([u_i](\underset{\sim}{\xi},T)\right) \frac{\partial}{\partial \xi_q} \overbrace{G_{np}(\underset{\sim}{x},t;\underset{\sim}{\xi},T)}^{G_{np}(\underset{\sim}{x},t-T;\underset{\sim}{\xi})} \nu_j \, d\Sigma_{\underset{\sim}{\xi}}$$

$$G_{np}(\underset{\sim}{x},t;\underset{\sim}{\xi},T) = G_{np}(\underset{\sim}{x},t-T;\underset{\sim}{\xi})$$

$$\boxed{u_n(\underset{\sim}{x},t) = \int_{\Sigma_{\underset{\sim}{\xi}}} \nu_j C_{ijpq} [u_i](\underset{\sim}{\xi},t) * \frac{\partial}{\partial \xi_q} G_{np}(\underset{\sim}{x},t;\underset{\sim}{\xi}) \, d\Sigma_{\underset{\sim}{\xi}}.}$$

$$\underset{\sim}{x} = (x_1, x_2, x_3) \longleftarrow u_n(\underset{\sim}{x}, t), \quad n = 1,2,3$$





$$\delta(\eta_1 - \xi_1)\,\delta(\eta_2 - \xi_2)\,\delta(\eta_3 - \xi_3)$$

$$\frac{\partial}{\partial \xi_q} G_{np}(\underset{\sim}{x}, t; \underset{\sim}{\xi}) = \int_{V_{\underset{\sim}{\eta}}} \delta(\underset{\sim}{\eta} - \underset{\sim}{\xi}) \frac{\partial}{\partial \eta_q} G_{np}(\underset{\sim}{x}, t; \underset{\sim}{\eta})\, dV_{\underset{\sim}{\eta}} =$$

$$= \cancelto{0}{\delta(\underset{\sim}{\eta} - \underset{\sim}{\xi})\, G_{np}}(\underset{\sim}{x}, t; \underset{\sim}{\eta}) \Big|_{\partial V} - \int_{V_{\underset{\sim}{\eta}}} \frac{\partial}{\partial \eta_q} \delta(\underset{\sim}{\eta} - \underset{\sim}{\xi})\, G_{np}(\underset{\sim}{x}, t; \underset{\sim}{\eta})\, dV_{\underset{\sim}{\eta}}$$

$$u_n(\underset{\sim}{x},t) = -\int_{\Sigma_{\underset{\sim}{\xi}}} \nu_j C_{ijpq}[u_i](\underset{\sim}{\xi},t) * \int_{V_{\underset{\sim}{\eta}}} \frac{\partial}{\partial \eta_q} \delta(\underset{\sim}{\eta}-\underset{\sim}{\xi})\, G_{np}(\underset{\sim}{x},t;\underset{\sim}{\eta})\, dV_{\underset{\sim}{\eta}} \cdot d\Sigma_{\underset{\sim}{\xi}}$$

$$= \int_{V_{\underset{\sim}{\eta}}} G_{np}(\underset{\sim}{x},t;\underset{\sim}{\eta}) * \left[-\int_{\Sigma_{\underset{\sim}{\xi}}} \frac{\partial}{\partial \eta_q} \delta(\underset{\sim}{\eta}-\underset{\sim}{\xi})\, \nu_j C_{ijpq}[u_i](\underset{\sim}{\xi},t)\, d\Sigma_{\underset{\sim}{\xi}} \right] dV_{\underset{\sim}{\eta}}$$

$$\forall \underset{\sim}{\eta} \qquad \boxed{e_p = -\int_{\Sigma_{\underset{\sim}{\xi}}} \frac{\partial}{\partial \eta_q} \delta(\underset{\sim}{\eta}-\underset{\sim}{\xi})\, \nu_j C_{ijpq}[u_i](\underset{\sim}{\xi},t)\, d\Sigma_{\underset{\sim}{\xi}} \qquad p = 1,2,3.}$$