MEM-284: Κυματική Διάδοση

Τμήμα Μαθηματικών και Εφαρμοσμένων Μαθηματικών, Πανεπιστήμιο Κρήτης

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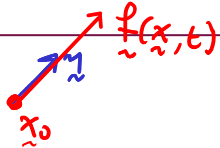
Εξίσωση Poisson

$$\nabla^2 \mathcal{G}(\mathbf{x}, t) = \delta(\mathbf{x})$$

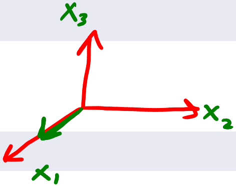
$$\mathcal{G}(\mathbf{x}, t) = -\frac{1}{4\pi r}, \quad r = \left[(x_1 - \underset{\substack{\shortparallel \\ 0}}{x_{01}})^2 + (x_2 - \underset{\substack{\shortparallel \\ 0}}{x_{02}})^2 + (x_3 - \underset{\substack{\shortparallel \\ 0}}{x_{03}})^2\right]^{1/2}$$



$$\mathbf{f}(\mathbf{x},t) = \mathbf{n}g(t)\delta(\mathbf{x}-\mathbf{x}_0)$$

Παράδειγμα : $\mathbf{n} = (1,0,0)$

$$\underset{\sim}{f} = \left(g(t)\delta(\underset{\sim}{x}-\underset{\sim}{x}_0), 0, 0\right)^T$$



$$\underset{\sim}{n} = \left(\delta_{1p}, \delta_{2p}, \delta_{3p}\right)$$

Μια αναπαράσταση με δυναμικά για την f

$$\underset{\sim}{f} = \alpha^2\nabla F + \beta^2\nabla\times\mathbf{Z}$$

$$\alpha^2 F = -\nabla\cdot\left(\frac{\mathbf{n}}{4\pi r}\right)g(t)$$

$$\beta^2\mathbf{Z} = \nabla\times\left(\frac{\mathbf{n}}{4\pi r}\right)g(t)$$

$$\nabla^2\left(-\frac{1}{4\pi r}\right) = \delta(\underset{\sim}{x}) \qquad r = \|x\|$$

$$\nabla^2 \underset{\sim}{u} = \nabla(\nabla\cdot\underset{\sim}{u}) - \nabla\times\nabla\times\underset{\sim}{u}$$

$$\underset{\sim}{\eta}\,\delta(\underset{\sim}{x}) = \nabla\left(\nabla\cdot\left(-\frac{\underset{\sim}{\eta}}{4\pi r}\right)\right) - \nabla\times\nabla\times\left(-\frac{\underset{\sim}{\eta}}{4\pi r}\right)$$

$$\underset{\sim}{\eta}\,\delta(\underset{\sim}{x})g(t) = \nabla\underbrace{\left(-\nabla\cdot\left(\frac{\eta}{4\pi r}\right)g(t)\right)}_{\alpha^2 F} + \nabla\times\underbrace{\left(\nabla\times\left(\frac{\underset{\sim}{\eta}}{4\pi r}\right)g(t)\right)}_{\beta^2 \underset{\sim}{Z}}$$

$$= \alpha^2\nabla F + \beta^2\nabla\times\underset{\sim}{Z}.$$

$$\nabla^2\phi - \alpha^{-2}\ddot{\phi} - \alpha^{-2}g(t)\nabla\cdot\left(\frac{\mathbf{n}}{4\pi r}\right) = 0$$

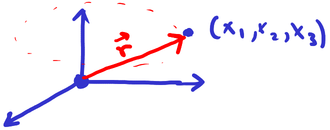
$$\nabla^2\boldsymbol{\psi} - \beta^{-2}\ddot{\boldsymbol{\psi}} + \beta^{-2}g(t)\nabla\times\left(\frac{\mathbf{n}}{4\pi r}\right) = \mathbf{0}$$

Βρίσκουμε Φ και Ψ βαθμωτά τ.ω

- Ορίζουμε $\phi = \nabla\cdot(\Phi\mathbf{n}), \quad \boldsymbol{\psi} = -\nabla\times(\Psi\mathbf{n})$

$$\nabla^2(\nabla\cdot\Phi\,\underset{\sim}{n}) - \alpha^{-2}\,\nabla\cdot(\ddot{\Phi}\,\underset{\sim}{n}) - \alpha^{-2}g(t)\,\nabla\cdot\left(\frac{\underset{\sim}{n}}{4\pi r}\right) = 0$$

$\phi(x_1, x_2, x_3, t)$



$$\nabla^2 \Phi - \alpha^{-2} \ddot{\Phi} - \alpha^{-2} g(t) \frac{1}{4\pi r} = 0$$

$$\nabla^2 \Psi - \beta^{-2} \ddot{\Psi} - \beta^{-2} g(t) \frac{1}{4\pi r} = 0$$

- Οι δύο εξισώσεις διαφέρουν μόνο στις ταχύτητες διάδοσης (α, β).

$$\Phi(\mathbf{x}, t; \mathbf{x}_0)$$

3D

$$\nabla^2\Phi - \alpha^{-2}\ddot{\Phi} - \alpha^{-2}g(t)\frac{1}{4\pi r} = 0$$

Η εξίσωση στις σφαιρικές συντεταγμένες

Άσκηση ∇^2

$\phi(\underset{\sim}{x}, t) \longrightarrow \phi(r,t)$

$$\Phi(r, t), \;\; r > 0$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial\Phi}{\partial r}) - \alpha^{-2}\ddot{\Phi} - \alpha^{-2}g(t)\frac{1}{4\pi r} = 0$$

$$\Phi(r, 0) = 0, \quad \dot{\Phi}(r, 0) = 0$$ $\longleftarrow$ Αρχικές συνθήκες.

$$\frac{\partial \Phi}{\partial r} = \frac{r\mathcal{X}_r - \mathcal{X}}{r^2}$$

$$r^2 \phi_r = r\mathcal{X}_r - \mathcal{X}$$

$$(r^2\phi')' = \mathcal{X}_r + r\mathcal{X}_{rr} - \mathcal{X}_r = r\mathcal{X}_{rr}$$

- Το πρόβλημα λύνεται πιο εύκολα με αλλαγή μεταβλητής

$$\frac{1}{r^2}(r^2\Phi')' = \frac{\mathcal{X}_{rr}}{r}$$

$$\ddot{\Phi} = \frac{\ddot{\mathcal{X}}}{r}$$

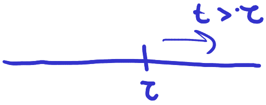
$$\Phi(r,t) = \frac{\mathcal{X}(r,t)}{r}, \quad r > 0$$

$$\mathcal{X}_{rr} - \alpha^{-2}\ddot{\mathcal{X}} - \alpha^{-2}\frac{g(t)}{4\pi} = 0$$

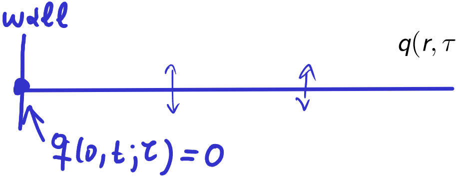
$$\mathcal{X}(r,0) = 0, \quad \dot{\mathcal{X}}(r,0) = 0$$

$\mathcal{X}_{rr} - \alpha^{-2}\ddot{\mathcal{X}} + f = 0$

$\mathcal{X}(r,0)=0$, $\dot{\mathcal{X}}(r,0)=0$.



$$q_{rr}(r,t;\tau) - \alpha^{-2}\ddot{q}(r,t;\tau) = 0$$



$$q(r,\tau;\tau) = 0, \quad \dot{q}(r,\tau;\tau) = -\frac{g(\tau)}{4\pi}$$

← Αρχικές συνθήκες (αρχή του χρόνου τ)

$$\mathcal{X}(r,t) = \int_0^t q(r,t;\tau)d\tau$$

- Το πρόβλημα είναι ορισμένο για $r > 0$
- Θα λύσουμε βρίσκοντας τη λύση για $r \in \mathbb{R}$ εισάγοντας επιπλέον τη συνοριακή συνθήκη $q(0,t;\tau) = 0$
- Στη συνέχεια θα περιορίσουμε τη λύση στο $r > 0$

Εάν τα αρχικά δεδομένα (αρχική μετατόπιση και ταχύτητα είναι περιττή συνάρτ.) Τότε και η λύση θα είναι περιττή συνάρτηση ως προς το r

$H(r) = \begin{cases} 1, & r > 0 \\ 0, & r \leq 0 \end{cases}$

Q - συμβολίζει την επέκταση σε όλο το $\mathbb{R}$

$f(-r) = -f(r)$

$r \in \mathbb{R}$

$$Q_{rr}(r,t;\tau) - \alpha^{-2}\ddot{Q}(r,t;\tau) = 0$$

$$Q(r,\tau;\tau) = 0, \quad \dot{Q}(r,\tau;\tau) = -\frac{g(\tau)}{4\pi}H(r) + \frac{g(\tau)}{4\pi}H(-r) = \begin{cases} -\frac{g(\tau)}{4\pi}, & r > 0 \\ 0, & r = 0 \\ \frac{g(\tau)}{4\pi}, & r < 0 \end{cases}$$

Λυσή για $r > \alpha(t-\tau)$

$$Q(r,t;\tau) = -\frac{1}{2\alpha}\int_{r-\alpha(t-\tau)}^{r+\alpha(t-\tau)} \frac{g(\tau)}{4\pi}d\xi = -\frac{g(\tau)}{4\pi}(t-\tau)$$

Λυσή για $r < \alpha(t-\tau)$

$$Q(r,t;\tau) = -\frac{1}{2\alpha}\int_{\alpha(t-\tau)-r}^{r+\alpha(t-\tau)} \frac{g(\tau)}{4\pi}d\xi = -\frac{g(\tau)}{\alpha 4\pi}r$$

$$-\frac{1}{2\alpha}\int_{r-\alpha(t-\tau)}^{r+\alpha(t-\tau)} \frac{g(\tau)}{4\pi}\, d\xi = -\frac{1}{\cancel{2\alpha}}\frac{g(\tau)}{4\pi}\Big[\underbrace{\cancel{r}+\alpha(t-\tau)\cancel{-r}+\alpha(t-\tau)}_{\cancel{2\alpha}(t-\tau)}\Big]$$

$$-\frac{1}{2\alpha}\int_{\alpha(t-r)-r}^{r+\alpha(t-r)} \frac{g(\tau)}{4\pi}\, d\xi = -\frac{g(\tau)}{\cancel{2}\alpha\cdot 4\pi}\Big[\underbrace{r+\alpha(t-\tau)-\alpha(t-\tau)+r}_{\cancel{2}r}\Big]$$

$q(r,t;\tau) = Q(r,t;\tau)\Big|_{r>0}$

Λυσή για $r > \alpha(t-\tau)$

$$q(r,t;\tau) = -\frac{1}{2\alpha}\int_{r-\alpha(t-\tau)}^{r+\alpha(t-\tau)} \frac{g(\tau)}{4\pi}\,d\xi = -\frac{g(\tau)}{4\pi}(t-\tau)$$

Λυσή για $0 < r < \alpha(t-\tau)$

$$q(r,t;\tau) = -\frac{1}{2\alpha}\int_{\alpha(t-\tau)-r}^{r+\alpha(t-\tau)} \frac{g(\tau)}{4\pi}\,d\xi = -\frac{g(\tau)}{\alpha 4\pi}r$$

$t-\tau-r/\alpha>0$

$\alpha(t-\tau)-r>0$

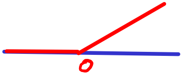
- Συνοπτικά γράφουμε

$$q(r,t;\tau) = \left(\frac{R(t-\tau-r/\alpha)}{4\pi} - \frac{t-\tau}{4\pi}\right)g(\tau)$$

$\frac{\cancel{t}-\cancel{\tau}-r/\alpha-\cancel{t}+\cancel{\tau}}{4\pi}$

$t-\tau-r/\alpha<0$

$-\frac{t-\tau}{4\pi}g(\tau)$



όπου $R(\xi) = \xi H(\xi) = \begin{cases} \xi, & \xi>0 \\ 0, & \xi\le 0 \end{cases}$

$$\chi(r,t) = \int_0^t \left(\frac{R(t-\tau-r/\alpha)}{4\pi} - \frac{t-\tau}{4\pi} \right) \delta(\tau)\, d\tau$$

$$\int f(t)\,\delta(t-t_0)\,dt = f(t_0)$$

$$\tau=0$$

$$= \frac{R(t-r/\alpha)}{4\pi} - \frac{t}{4\pi}.$$

$$\Phi = \frac{\chi}{r}$$

Για $g(t) = \delta(t)$ έχουμε

$$\xi(r,t) = \frac{R(t-r/\alpha)}{4\pi} - \frac{t}{4\pi}$$

Επομένως

$$\boxed{\Phi(r,t) = \frac{R(t-r/\alpha)}{4\pi r} - \frac{t}{4\pi r}, \quad r > 0}$$

Ομοία

$$\boxed{\Psi(r,t) = \frac{R(t-r/\beta)}{4\pi r} - \frac{t}{4\pi r}, \quad r > 0}$$

$\varphi = \nabla \cdot (\Phi \underset{\sim}{n})$ $\quad \underset{\sim}{\psi} = -\nabla \times (\Psi \underset{\sim}{n})$

$H(t) = \begin{cases} 1, & t>0 \\ 0, & t \leq 0 \end{cases}$

$G_{ip} = \nabla \varphi + \nabla \times \underset{\sim}{\psi}$

$$\mathcal{G}_{ip} = \frac{\gamma_i\gamma_p}{4\pi\rho\alpha^2 r}\delta(t - r/\alpha) + \frac{(\delta_{ip} - \gamma_i\gamma_p)}{4\pi\rho\beta^2 r}\delta(t - r/\beta)$$

$R(t) = t\,H(t)$

$G_{ip}(\underset{\sim}{x}, t)$

$$\ast \begin{cases} -\dfrac{(\delta_{ip} - 3\gamma_i\gamma_p)}{4\pi\rho\alpha^2}\left[\dfrac{\alpha}{r^2}H(t - r/\alpha) + \dfrac{\alpha^2}{r^3}R(t - r/\alpha)\right] \\ +\dfrac{(\delta_{ip} - 3\gamma_i\gamma_p)}{4\pi\rho\beta^2}\left[\dfrac{\beta}{r^2}H(t - r/\beta) + \dfrac{\beta^2}{r^3}R(t - r/\beta)\right] \end{cases}$$

$A = \dfrac{\delta_{ip} - 3\gamma_i\gamma_p}{4\pi\rho}$

όπου $\gamma_i = x_i/r$

$t > r/\alpha$.

$$\frac{\alpha}{r^2}H(t - r/\alpha) + \frac{\alpha^2}{r^3}R(t - r/\alpha) = \frac{\alpha}{r^2} + \frac{\alpha^2}{r^3}(t - r/\alpha) = \frac{\alpha^2}{r^3}t$$

$$= \begin{cases} \dfrac{\alpha}{r^2} + \dfrac{\alpha^2}{r^3}\cdot(t - r/\alpha) = \dfrac{\alpha}{r^2} + \dfrac{\alpha^2}{r^3}t - \dfrac{\alpha}{r^2} = \dfrac{\alpha^2}{r^3}t, & t > r/\alpha \\ 0, & \text{διαφορετικά} \end{cases}$$

$t > r/\alpha \qquad t > r/\beta$

$(*) \quad -A \cdot \frac{t}{r^3} + A \frac{t}{r^3}$

$-A \frac{\cancel{r}}{r^{\cancel{3}}\alpha} = -\frac{A}{r^2\alpha}$

$-A/r^2\alpha$

Τελικά η συνάρτηση Green γράφεται

$$\mathcal{G}_{ip}(\mathbf{x}, t) = \frac{\gamma_i\gamma_p}{4\pi\rho\alpha^2 r}\delta(t - r/\alpha) + \frac{(\delta_{ip} - \gamma_i\gamma_p)}{4\pi\rho\beta^2 r}\delta(t - r/\beta)$$
$$- \frac{(\delta_{ip} - 3\gamma_i\gamma_p)}{4\pi\rho r^3} tH(t, r/\alpha, r/\beta)$$

$-A/r^2\beta$

$\delta_{ip} = \begin{cases} 1, & i = p \\ 0, & i \neq p. \end{cases}$

$H(t, r/\alpha, r/\beta) = \begin{cases} 1, & t \in [r/\alpha, r/\beta) \\ 0, & \text{διαφορετικά} \end{cases}$

$r = 10.000$ m

Λύση μακρινού πεδίου (far field) $r >> 0$

$$\mathcal{G}_{ip} = \mathcal{G}_{ip}^{P} + \mathcal{G}_{ip}^{S}$$

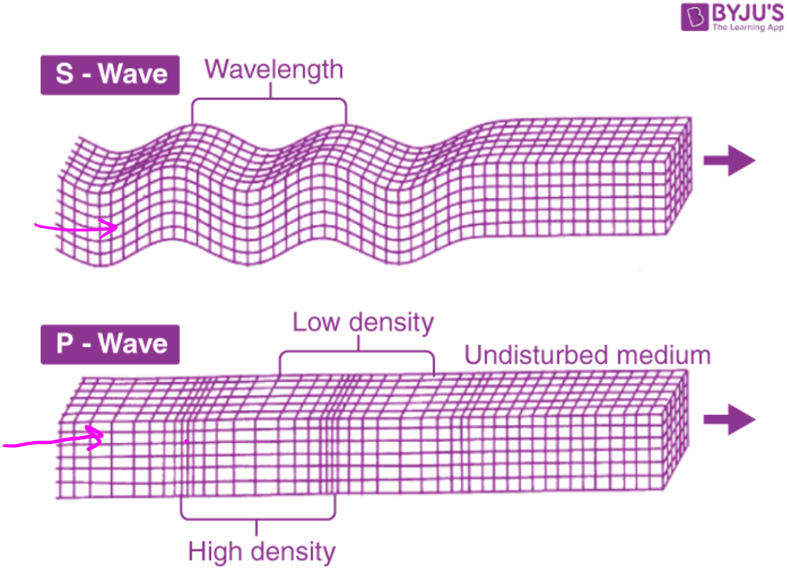
P-wave Primary

$\gamma_i = \frac{x_i}{r}$

$$\mathcal{G}_{ip}^{P} = \frac{\gamma_i \gamma_p}{4\pi\rho\alpha^2 r}\delta(t - r/\alpha)$$

S-wave Secondary

$$\mathcal{G}_{ip}^{S} = \frac{(\delta_{ip} - \gamma_i \gamma_p)}{4\pi\rho\beta^2 r}\delta(t - r/\beta)$$


BYJU'S
The Learning App
S - Wave
Wavelength
P - Wave
Low density
Undisturbed medium
High density