MEM-284: Κυματική Διάδοση

Τμήμα Μαθηματικών και Εφαρμοσμένων Μαθηματικών, Πανεπιστήμιο Κρήτης

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11η Διάλεξη - 8.4.2022

$u_n = \int \mathcal{G} * f$

Θεώρημα αναπαράστασης

$$u_n(\mathbf{x}, t) = \int_{\Sigma_\xi} \nu_j c_{ijpq}[u_i](\boldsymbol{\xi}, t) * \frac{\partial}{\partial \xi_q} \mathcal{G}_{np}(\mathbf{x}, t; \boldsymbol{\xi}) d\Sigma_\xi$$

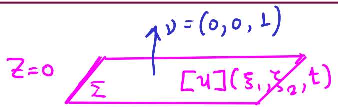
Ισοδύναμη δύναμη

$$\boxed{e_p(\boldsymbol{\eta}, t)} = -\int_{\Sigma_\xi} \frac{\partial}{\partial \eta_q} \delta(\boldsymbol{\eta} - \boldsymbol{\xi}) \nu_j c_{ijpq}[u_i](\boldsymbol{\xi}, t) d\Sigma_\xi$$

$\eta \in \Sigma$

Ομοιόμορφα και ισοτροπικά ελαστικά μέσα

$$c_{ijpq} = \lambda \delta_{ij} \delta_{pq} + \mu(\delta_{ip} \delta_{jq} + \delta_{iq} \delta_{jp})$$



$\cdot\ u(\underset{\sim}{x},t)$

Μετατόπιση

$$\mathcal{G}_{np}(x,t;\xi) * e_p(\xi,t) = \int_{\mathbb{R}} \mathcal{G}_{np}(\underset{\sim}{x},t-\tau;\underset{\sim}{\xi})\, e_p(\underset{\sim}{\xi},\tau)\,d\tau$$

$$\| $$

$$\mathcal{G}_{np}(\underset{\sim}{x},t;\xi,\tau)$$

$$u_n(\mathbf{x},t) = \int_{V_0} \mathcal{G}_{np}(\mathbf{x},t;\boldsymbol{\xi}) * e_p(\boldsymbol{\xi},t)\,dV_{\boldsymbol{\xi}}$$

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \qquad \underbrace{d\xi_1\, d\xi_2}_{d\Sigma}$$

- V_0 : όγκος στον οποίο η $\mathbf{e} = (e_1, e_2, e_3)$ είναι μη μηδενική.

Από την άσκηση 3 της διάλεξης 10 είχαμε $\quad z=0$

$$\mathbf{e} = \left(-\mu H(t)\delta(\eta_1)\delta(\eta_2)\frac{\partial}{\partial\eta_3}\delta(\eta_3),\ 0,\ -\mu H(t)\delta(\eta_2)\delta(\eta_3)\frac{\partial}{\partial\eta_1}\delta(\eta_1)\right)$$

$$u_n(\underset{\sim}{x},t) = \int_{\Sigma}\int_{\mathbb{R}} \mathcal{G}_{np}(\underset{\sim}{x},t;\underset{\sim}{\xi},\tau)\, e_p(\underset{\sim}{\xi},\tau)\,d\tau\, dV_{\underset{\sim}{\xi}} = -\mu\int_{\Sigma}\int_{\mathbb{R}} \mathcal{G}_{n1}(\underset{\sim}{x},t,\underset{\sim}{\xi},\tau)\, H(\tau)\,\delta(\xi_1)\,\delta(\xi_2)\,\frac{\partial}{\partial\xi_3}\delta(\xi_3)\, d\tau\, d\Sigma_{\xi}$$

$$-\mu\int_{\Sigma}\int_{\mathbb{R}} \mathcal{G}_{n3}(\underset{\sim}{x},t,\underset{\sim}{\xi},\tau)\, H(\tau)\,\delta(\xi_2)\,\delta(\xi_3)\,\frac{\partial}{\partial\xi_1}\delta(\xi_1)\, d\tau\, d\Sigma_{\underset{\sim}{\xi}}$$

$$= \mu \frac{\partial \delta(\xi_3)}{\partial \xi_3} \int_{\Sigma} \int_0^{\infty} g_{n1}(\underset{\sim}{x}, t; \underset{\sim}{\xi}, \tau)\, \delta(\xi_1)\delta(\xi_2)\, d\tau\, d\Sigma_{\underset{\sim}{\xi}} \underbrace{- \mu\, \delta(\xi_3) \int_{\Sigma} \int_0^{\infty} g_{n3}\, \delta(\xi_2) \frac{\partial}{\partial \xi_1} \delta(\xi_1)\, d\tau\, d\Sigma_{\xi}}_{*}$$

$$= -\mu \frac{\partial}{\partial \xi_3} \delta(\xi_3) \int_0^{\infty} g_{n1}(\underset{\sim}{x}, t; (0,0,\xi_3), \tau)\, d\tau + *$$

$$* = -\mu\, \delta(\xi_3) \int_0^{\infty} d\tau \int_{\Sigma} g_{n3}(\underset{\sim}{x}, t; \underset{\sim}{\xi}, \tau)\, \delta(\xi_2) \frac{\partial}{\partial \xi_1} \delta(\xi_1)\, d\Sigma_3 =$$

$$= \mu\, \delta(\xi_3) \int_0^{\infty} d\tau \frac{\partial}{\partial \xi_1} g_{n3}(\underset{\sim}{x}, t; (0,0,\xi_3), \tau)$$

Προσέγγιση μη σημειακής ασυνέχειας με σημειακή πηγή.

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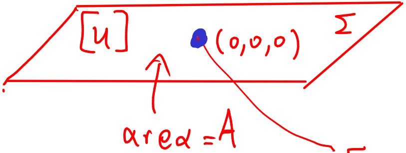
$$[u_1(\xi_1, \xi_2, t)],\ [u_2] = [u_3] = 0, \quad (\xi_1, \xi_2) \in \Sigma$$



$$\overline{[u]}(t) = \frac{1}{A}\int_\Sigma [u_1](\xi_1, \xi_2, t)\, d\xi_1 d\xi_2$$

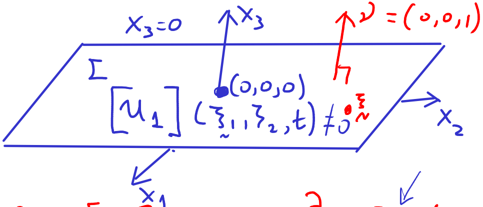
$$[u_1(\xi_1, \xi_2, t)] = \frac{1}{\mu}\underbrace{\mu\, \overline{[u]}(t)\, A}_{M_0(t)}\ \delta(\xi_1)\delta(\xi_2)$$

$M_0(t) \leftarrow$ ροπή της ισοδύναμης δύναμης

$$= \frac{1}{\mu} M_0(t)\, \delta(\xi_1)\delta(\xi_2)$$

$u_n(\underset{\sim}{x}, t) = ;$

example:



$\bullet\ \underset{\sim}{x}$

$[u_2] = [u_3] = 0$

$$u_n(\underset{\sim}{x}, t) = \int_{\Sigma_\xi} \nu_j C_{ijpq} [u_i](\underset{\sim}{\xi}, t) * \frac{\partial}{\partial \xi_q} G_{np}(\underset{\sim}{x}, t; \underset{\sim}{\xi}) \, d\Sigma_{\underset{\sim}{\xi}}$$

$i = 1 \quad j = 3$

$p = 1, q = 3$

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$p = 3, q = 1.$

$C_{13pq} = \mu$, αν $p = 1, q = 3$ ή $p = 3, q = 1$.

$$u_n(\underset{\sim}{x}, t) = \int_{\Sigma_\xi} \mu [u_1](\underset{\sim}{\xi}, t) * \left\{ \frac{\partial}{\partial \xi_1} G_{n3}(\underset{\sim}{x}, t; \underset{\sim}{\xi}) + \frac{\partial}{\partial \xi_3} G_{n1}(\underset{\sim}{x}, t; \underset{\sim}{\xi}) \right\} d\Sigma_{\underset{\sim}{\xi}}$$

$G_{np} = \underline{G^P_{np}} + G^S_{np}$, $\quad G^P_{np} \xrightarrow{\theta.\alpha\nu\alpha\pi.} u^P_n$, $\quad G^S_{np} \longrightarrow u^S_n$

$u_n = u^P_n + u^S_n$

$f * (g_1 + g_2) = f * g_1 + f * g_2 \qquad \int g_1 + g_2 = \int g_1 + \int g_2$

Μελέτη για τα p-waves

$$G_{np}^{P} = \frac{\gamma_p}{4\pi\rho\alpha^2}\frac{1}{r}\gamma_n\,\delta(t - r/\alpha), \quad \gamma_i = \frac{x_i - \xi_i}{r}$$

$$r = \left((x_1-\xi_1)^2 + (x_2-\xi_2)^2 + (x_3-\xi_3)^2\right)^{1/2}$$

$$\frac{\partial \gamma_i}{\partial \xi_q} = \frac{\partial}{\partial \xi_q}\left(\frac{x_i - \xi_i}{r}\right) = \frac{\frac{\partial}{\partial \xi_q}(x_i - \xi_i)\, r - (x_i - \xi_i)\frac{\partial}{\partial \xi_q} r}{r^2} =$$

$$= \frac{-\delta_{iq}\, r - (x_i - \xi_i)\frac{\partial}{\partial \xi_q} r}{r^2} \quad (*)$$

$$\frac{\partial}{\partial \xi_q}\overset{0}{\cancel{x_i}} - \frac{\partial \xi_i}{\partial \xi_q} = \delta_{iq}$$

$$\frac{\partial r}{\partial \xi_q} = \frac{\partial}{\partial \xi_q}\left\{\left((x_1-\xi_1)^2 + (x_2-\xi_2)^2 + (x_3-\xi_3)^2\right)^{1/2}\right\} = -\frac{1}{\cancel{2}} r^{-1}\cdot\cancel{2}(x_q - \xi_q) = -\frac{x_q - \xi_q}{r} = -\gamma_q$$

$$(*) = -\frac{\delta_{iq}}{r} - \frac{1}{r}\frac{x_i - \xi_i}{r}\cdot(-\gamma_q) = \frac{-\delta_{iq} + \gamma_i\gamma_q}{r}$$

$$\frac{\partial}{\partial \xi_q}\delta(t - r/\alpha) = \delta'(t - r/\alpha)\cdot\frac{\partial}{\partial \xi_q}(t - r/\alpha) = -\frac{1}{\alpha}\delta'(t - r/\alpha)\frac{\partial}{\partial \xi_q} r = \frac{\gamma_q}{\alpha}\delta'(t - r/\alpha)$$

$$\frac{\partial G^{p}_{n1}}{\partial \xi_3} = \frac{\gamma_1}{4\pi\rho\alpha^2}\frac{\partial}{\partial \xi_3}\left\{\underbrace{\frac{1}{r}\gamma_n\,\delta(t - r/\alpha)}\right\} \qquad r \gg 1$$

$$\frac{\partial}{\partial \xi_3}\left\{\frac{1}{r}\right\}\gamma_n\,\delta(t-r/\alpha) + \frac{1}{r}\frac{\partial \gamma_n}{\partial \xi_3}\delta(t-r/\alpha) + \frac{1}{r}\gamma_n\frac{\partial}{\partial \xi_3}\delta(t-r/\alpha)$$

$$= \frac{\gamma_3}{r^2}\gamma_n\,\delta(t-r/\alpha) + \frac{-\delta_{n3}+\gamma_n\gamma_3}{r^2}\delta(t-r/\alpha) + \frac{1}{r}\gamma_n\frac{\gamma_3}{\alpha}\delta'(t-r/\alpha)$$

$$\overset{r\gg 1}{\approx} \frac{1}{r\alpha}\gamma_n\gamma_3\,\delta'(t-r/\alpha).$$

$$\frac{\partial G^{p}_{n1}}{\partial \xi_3} \approx \frac{\gamma_n\gamma_1\gamma_3}{4\pi\rho r\alpha^3}\delta'(t-r/\alpha) \qquad \text{όμοια} \qquad \frac{\partial G^{p}_{n3}}{\partial \xi_1} \approx \frac{\gamma_n\gamma_1\gamma_3}{4\pi\rho r\alpha^3}\delta'(t-r/\alpha)$$

$p = 1,\ q = 3$

$$\left\{\frac{\partial G^{p}_{n1}}{\partial \xi_3} + \frac{\partial G^{p}_{n3}}{\partial \xi_1}\right\} \approx \frac{\gamma_n\gamma_1\gamma_3}{2\pi\rho r\alpha^3}\delta'(t-r/\alpha)$$

$r >> 1$

$$u^{f}_{n}(\underset{\sim}{x}, t) = \mu A \int_{\Sigma_{\underset{\sim}{\xi}}} [\overline{u_1}](t)\, \underline{\delta(\xi_1)\delta(\xi_2)} * \frac{\gamma_n \gamma_1 \gamma_3}{2\pi\rho r \alpha^3} \delta'(t - r/\alpha)\, d\Sigma_{\underset{\sim}{\xi}} =$$

$$= \mu A \frac{\gamma_n \gamma_1 \gamma_3}{2\pi \rho r \alpha^3} [\overline{u_1}](t) * \delta'(t - r/\alpha)$$

$r = \|x\|$

$\gamma_i = \frac{x_i}{\|x\|}$

$$[\overline{u_1}](t) * \delta'(t - r/\alpha) = \int_{\mathbb{R}} [\overline{u_1}](\tau)\, \delta'(t - \tau - r/\alpha)\, d\tau =$$

$$f(t) * g(t) = \int_{\mathbb{R}} f(\tau)\, g(t-\tau)\, d\tau = \int_{\mathbb{R}} g(\tau)\, f(t-\tau)\, d\tau = g(t) * f(t)$$

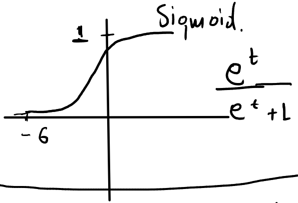
$$= \cancel{[\overline{u_1}](\tau)\, \delta(t - \tau - r/\alpha)\Big|_{-\infty}^{+\infty}}^{\,0} - \int_{\mathbb{R}} [\overline{u_1}]'(\tau)\, \delta(t - \tau - r/\alpha)\, d\tau$$

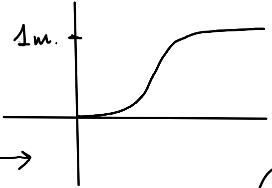
$$= -\int_{\mathbb{R}} [\overline{u_1}]'(\tau)\,\delta(t-\tau-r/\alpha)\,d\tau = -[\overline{u_1}]'(t-r/\alpha)$$

$$u_\eta^P(\underset{\sim}{x},t) = -\mu A \frac{\gamma_\eta \gamma_1 \gamma_3}{2\pi\rho r \alpha^3} [\overline{u_1}]'(t-r/\alpha), \quad \eta=1,2,3$$

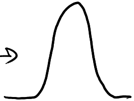
Παράδειγμα

$$[\overline{u_1}](t) = \frac{e^{t-6}}{e^{t-6}+1}$$



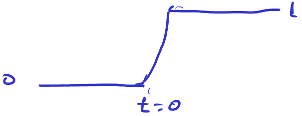


$$[\overline{u_1}]'(t) = \frac{e^{t-6}(e^{t-6}+1) - e^{t-6}\,e^{t-6}}{(e^{t-6}+1)^2} = \frac{e^{t-6}}{(e^{t-6}+1)^2}$$



$$u_n^P(\underset{\sim}{x},t) = -\mu A \frac{\gamma_n\gamma_1\gamma_3}{2\pi\rho r\alpha^3}\,\frac{e^{t-6-r/\alpha}}{\left(e^{t-6-r/\alpha}+1\right)^2}$$

$$\mu = \rho b^2$$



Παράδειγμα:

$$[\bar{u}](t) = H(t),\quad t\in\mathbb{R}$$

θ.δ.ο

$$H'(t) = \delta(t)$$

$$[\bar{u}]'(t) = \delta(t)$$

$$H(t) = \int_{-\infty}^{t} H'(\tau)\,d\tau = \begin{cases} 1, & t>0 \\ 0, & t<0 \end{cases}$$

$$\int_{-\infty}^{t}\delta(\tau)\,d\tau = H(t)$$

$$u_n^P(\underset{\sim}{x},t) \overset{t}{\propto} \delta(t - r/\alpha)$$

$$\gamma_i = \frac{x_i}{r} \qquad r = (x_1^2 + x_2^2 + x_3^2)^{1/2}$$

$$g(x_i, r)$$

$$u_1(\underset{\sim}{x}, t) = u_1(\underline{x_1, x_2}, x_3, \underline{t})$$