MEM-284: Κυματική Διάδοση

Τμήμα Μαθηματικών και Εφαρμοσμένων Μαθηματικών, Πανεπιστήμιο Κρήτης

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1ο εργαστήριο ασκήσεων - 11.3.2022

$$\int_{-\infty}^{+\infty} f(x)\,\delta(x-x_0)\,dx = f(x_0)$$

$$H(x)=\begin{cases}1, & x>0\\ 0, & x\le 0\end{cases}$$

Λύστε την παρακάτω κυματική εξίσωση

$$u_{xx} - \underbrace{\frac{1}{4}}_{c^{-2}} u_{tt} + \overbrace{\delta(x+3)(\delta(t)+\delta(t-1))}^{f(x,t)} = 0, \quad x\in\mathbb{R}, t\in\mathbb{R}$$

$$u(x,0) = \overset{F(x)}{\sin(x)}, \quad u_t(x,0) = \overset{G(x)}{H(x)}$$

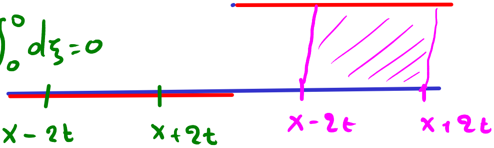
$$u(x,t) = \overset{1}{\frac{F(x-ct)+F(x+ct)}{2}} + \underset{2}{\frac{1}{2c}\int_{x-ct}^{x+ct} G(\xi)\,d\xi} + \overset{3}{\frac{c}{2}\int_0^t \int_{x-c(t-\tau)}^{x+c(t-\tau)} f(\xi,\tau)\,d\xi\,d\tau}$$

1 $\dfrac{\sin(x-2t)+\sin(x+2t)}{2}$

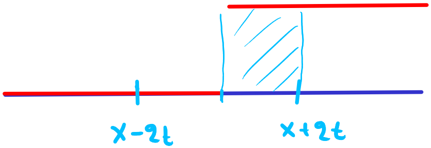
2 $\dfrac{1}{4}\displaystyle\int_{x-2t}^{x+2t} H(\xi)\,d\xi =$

$$= \frac{1}{4}\int_{\max(0,\,x-2t)}^{\max(0,\,x+2t)} d\xi = \frac{1}{4}\Big[\max(0,\,x+2t) - \max(0,\,x-2t)\Big]$$

$$\int_0^0 d\xi = 0$$



$$x+2t-(x-2t) = 4t$$



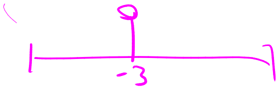
$$f(x,t) = \delta(x+3)\big(\delta(t) + \delta(t-1)\big)$$

-3

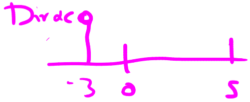
$$\int_{x-2(t-\tau)}^{x+2(t-\tau)} \delta(\xi+3)\big(\delta(\tau)+\delta(\tau-1)\big)\, d\xi =$$

$$= \big(\delta(\tau)+\delta(\tau-1)\big) \underbrace{\int_{x-2(t-\tau)}^{x+2(t-\tau)} \delta(\xi+3)\, d\xi}_{\mathbb{1}\{-3 \in [x-2(t-\tau),\, x+2(t-\tau)]\}}$$

$$\int_{-\infty}^{+\infty} \delta(\xi+3)\, d\xi = 1$$



$$\mathbb{1}\{\text{συνθηκη}\} = \begin{cases} 1, & \text{συνθήκη ισχύει} \\ 0, & \text{διαφορετικά} \end{cases}$$



$$\int_0^t \left(\delta(\tau)+\delta(\tau-1)\right) \mathbb{1}\left\{-3 \in \left[x-2(t-\tau),\, x+2(t-\tau)\right]\right\} d\tau =$$

$$\int_0^t \delta(\tau)\, \mathbb{1}\left\{-3 \in \left[\overset{\tau}{\cdots}\right]\right\} d\tau + \int_0^t \delta(\tau-1)\, \mathbb{1}\left\{-3 \in \left[\overset{\tau}{\cdots}\right]\right\} d\tau =$$

$$= \mathbb{1}\left\{-3 \in \left[x-2t,\, x+2t\right]\right\} + \mathbb{1}\left\{-3 \in \left[x-2t+2,\, x+2t-2\right]\right\}$$

Θεωρώντας αρμονικές λύσεις, λύστε την εξίσωση

$$u_t + u_x - u_{xx} = 0 \qquad \omega(k)$$

$$u = A e^{i(kx - \omega t)}$$

$$k^2 + ik - i\omega = 0$$

$$\frac{k^2}{i} = \frac{ik^2}{i^2}$$

$$i\omega = k^2 + ik$$

$$\omega(k) = \frac{k^2}{i} + k = k - ik^2$$

$$u(x,t) = A e^{i(kx - kt) - k^2} = A e^{-k^2 t} e^{i(kx - kt)}$$

μοναδιαία ταχύτητα $c = \frac{k}{k} = 1.$

Άσκηση 2

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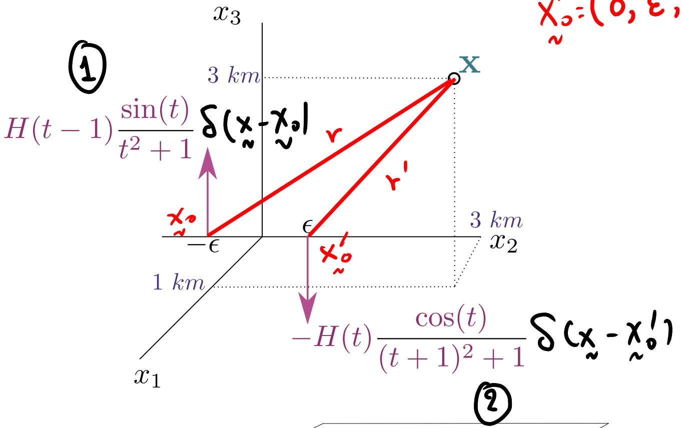
Υπολογίστε την μετατόπιση στο σημείο x για $t > 0$

$$\rho = 2500\ kg/m^3,\ \alpha = 6000\ m/s$$

Σημεία

$\underset{\sim}{x}_0 = (0, -\varepsilon, 0)$

$\underset{\sim}{x}_0' = (0, \varepsilon, 0)$



$$u_i(\underset{\sim}{x},t) = \int_0^t d\tau \underbrace{\int_V f_3(\underset{\sim}{\xi},\tau)\, \mathcal{G}_{i3}(\underset{\sim}{x}, t-\tau; \underset{\sim}{\xi})}_{I} dV(\underset{\sim}{\xi})$$

1

$$I = H(\tau-1)\frac{\sin\tau}{\tau^2+1}\int_V \delta(\underset{\sim}{\xi}-\underset{\sim}{x_0})\, \mathcal{G}_{i3}(\underset{\sim}{x}, t-\tau); \underset{\sim}{\xi})\, dV(\underset{\sim}{\xi}) =$$

$$= H(\tau-1)\frac{\sin\tau}{\tau^2+1}\mathcal{G}_{i3}(\underset{\sim}{x}, t-\tau; \underset{\sim}{x_0})$$

$$\mathcal{G}_{i3}(\underset{\sim}{x}, t) = \frac{\gamma_i\gamma_3}{4\pi\rho\alpha^2}\frac{1}{r}\delta(t - r/\alpha)$$

$$G_{i3}(\underset{\sim}{x}, t; \underset{\sim}{x_0}) = \frac{\gamma_i \gamma_3}{4\pi\rho\alpha^2} \frac{1}{r} \delta(t - r/\alpha) \qquad \gamma_i = \frac{x_i - x_{0i}}{r}$$

$$r = \|\underset{\sim}{x} - \underline{x}_0\|$$

σταθερά ως προς τ

$$\int_0^t \frac{\gamma_i \gamma_3}{4\pi\rho\alpha^2} \frac{1}{r} \delta(t - \tau - r/\alpha) H(\tau - 1) \frac{\sin\tau}{\tau^2+1} \, d\tau =$$

$$= \frac{\gamma_i \gamma_3}{4\pi\rho\alpha^2} \frac{1}{r} H\left(t - r/\alpha - 1\right) \frac{\sin(t - r/\alpha)}{(t - r/\alpha)^2 + 1}$$

Άσκηση 3

Δείξτε ότι $(f * g)(t) = (g * f)(t)$ θ.ν.δ.ο

$\tau^* = t - \tau \rightarrow \tau = t - \tau^*$

$$\int_{-\infty}^{+\infty} f(\tau)\, g(\overbrace{t-\tau}^{\tau^*})\, d\tau = \int_{-\infty}^{+\infty} g(\tau)\, f(t-\tau)\, d\tau$$

$d\tau^* = -d\tau$

$$\int_{+\infty}^{-\infty} f(t-\tau^*)\, g(\tau^*)\, (-d\tau^*) = \int_{-\infty}^{+\infty} g(\tau^*)\, f(t-\tau^*)\, d\tau^* =$$

$$= g * f$$

$f * g \qquad F(\omega)G(\omega) = G(\omega)F(\omega) \longrightarrow g * f$

Άσκηση 4

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