MEM-264 Applied Statistics

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$$\phi(x) = \begin{cases} 1, & t(x) \in C_\alpha^{(1)} \quad \text{(Reject } H_0 \text{ with probability } 1) \\ \gamma, & t(x) \in C_\alpha^{(\gamma)} \quad \text{(Reject } H_0 \text{ with probability } \gamma \in (0,1)) \\ 0, & \text{otherwise} \quad \text{(Reject } H_0 \text{ with probability } 0) \equiv \text{(Do not Reject } H_0) \end{cases}$$

Power function of a test ϕ

$\Gamma \sim U[0,1]$ observation : γ^*

$$w(\theta) = P\{\text{Reject } H_0\} = \mathbb{E}\{\phi(X)\}$$

- A good is one that makes $w(\theta)$ as large as possible on Θ_1, while satisfying $w(\theta) \leq \alpha$ for all $\theta \in \Theta_0$.

$$P\{\text{Reject } H_0\} = P\{t(x) \in C_\alpha^{(1)}\} + P\{t(x) \in C_\alpha^{(\gamma)}\} \cdot \gamma$$

$$P\{\text{Reject } H_0\} = P\{t(x) \in C_\alpha^{(1)}\} + P\{t(X) \in C_\alpha^{(\gamma)}, \Gamma \leq \gamma\} =$$

$$= P\{t(x) \in C_\alpha^{(1)}\} + P\{t(X) \in C_\alpha^{(\gamma)}\} \underbrace{P\{\Gamma \leq \gamma\}}_{\gamma} =$$

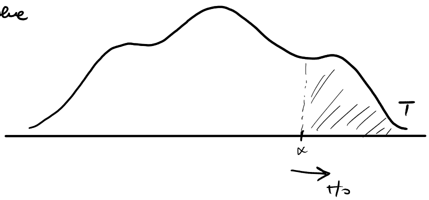
$$= 1 \cdot P\{t(x) \in C_\alpha^{(1)}\} + \gamma P\{t(x) \in C_\alpha^{(\gamma)}\} + 0 \cdot P\{t(x) \in (C_\alpha^{(1)} \cup C_\alpha^{(\gamma)})^c\}$$

$$= \mathbb{E}\{\phi(X)\}$$

Errors of type I & II

- **Error of type I**: Reject H_0, but $\theta \in \Theta_0$

For example
if H_0 : simple



- **Error of type II**: Do not reject H_0, but $\theta \in \Theta_1$.

$$f(x;\theta_0) \qquad\qquad f(x;\theta_1)$$

$$H_0 : \theta = \theta_0 \quad \text{vs} \quad H_1 : \theta = \theta_1$$

We define the **likelihood ratio** $\Lambda(x)$ by

$$\Lambda(x) = \frac{f(x;\theta_1)}{f(x;\theta_0)}$$

Definition : Likelihood Ratio Test

$$\phi_0(x) = \begin{cases} 1, & \Lambda(x) > K \\ \gamma(x), & \Lambda(x) = K \\ 0, & \Lambda(x) < K \end{cases}$$

where $K \geq 0$ is a constant and $\gamma(x)$ a function such that $0 \leq \gamma(x) \leq 1$ for all x.

Neyman-Pearson theorem

(α)
- Optimality

For any K and $\gamma(x)$, the test ϕ_0 has maximum power among all test whose sizes are no greater than the size of ϕ_0.

(β)
- Existence

Given $\alpha \in (0, 1)$, there exist constants K and γ_0 s.t the Likelihood Ratio Test has size exactly α.

(γ)
- Uniqueness

If the test ϕ has size α and is of maximum power amongst all possible tests of size α, then ϕ is a likelihood ratio test, except possibly on a set of zero probability under both H_0 and H_1.

Neyman-Pearson theorem : Proof

Size:

α) $H_0 : \theta = \theta_0$ $\qquad f(x;\theta_0)$ $\qquad \phi = \begin{cases} >0 & \text{Reject. with probability} \\ 0 & \text{Do not reject} \end{cases}$ $\qquad (K, \gamma(x))$

$\mathbb{E}_{\theta_0}\{\phi\} = \alpha$

$\downarrow$

Let ϕ be an arbitrary test function for which $\mathbb{E}_{\theta_0}\{\phi(x)\} \leq \mathbb{E}_{\theta_0}\{\phi_0(x)\}$ $(*)$

Let's define $v(x) = \big(\phi_0(x) - \phi(x)\big)\big(f(x;\theta_1) - K f(x;\theta_0)\big)$

* if $\Lambda(x) > K \Leftrightarrow f(x;\theta_1) - K f(x;\theta_0) > 0$ then $\phi_0(x) = 1 \Rightarrow \phi_0(x) - \phi(x) \geq 0 \Rightarrow v(x) \geq 0$
* if $\Lambda(x) < K \Leftrightarrow f(x;\theta_1) - K f(x;\theta_0) < 0$ then $\phi_0(x) = 0 \Rightarrow \phi_0(x) - \phi(x) \leq 0 \Rightarrow v(x) \geq 0$
* if $\Lambda(x) = K \Rightarrow f(x;\theta_1) - K f(x;\theta_0) = 0$ then $v(x) = 0$

for any x we have $v(x) \geq 0$.

$$0 \leq \int_{-\infty}^{+\infty} v(x)\,dx = \int_{\mathbb{R}} \phi_0(x) f(x;\theta_1)\,dx - \int_{\mathbb{R}} \phi(x) f(x;\theta_1)\,dx + K\left\{\int_{\mathbb{R}} \phi(x) f(x;\theta_0)\,dx - \int_{\mathbb{R}} \phi_0(x) f(x;\theta_0)\,dx\right\} =$$

$$= \mathbb{E}_{\theta_1}\{\phi_0(x)\} - \mathbb{E}_{\theta_1}\{\phi(x)\} + K\underbrace{\left\{\mathbb{E}_{\theta_0}\{\phi(x)\} - \mathbb{E}_{\theta_0}\{\phi_0(x)\}\right\}}_{\leq 0}$$

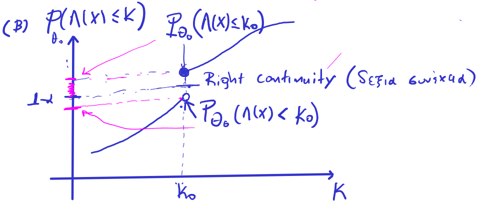
therefore $\qquad \mathbb{E}_{\theta_1}\{\phi_0(x)\} \geq \mathbb{E}_{\theta_1}\{\phi(x)\}$ $(**)$

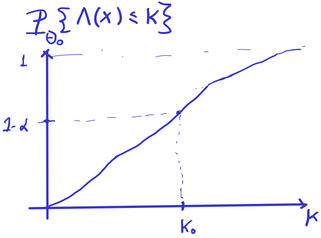
$(*), (**) \Rightarrow \quad \mathbb{E}_{\theta}\{\phi_0(X)\} \geq \mathbb{E}_{\theta}\{\phi(X)\}$

$W_{\phi_0}(\theta) \geq W_{\phi}(\theta)$

under H_0

(B)





$\exists K_0$ such that $P_{\theta_0}\{\Lambda(X) \leq K_0\} = 1-\alpha$

$\Downarrow$

$P_{\theta_0}\{\Lambda(X) > K_0\} = \alpha$

Let $\gamma = 0$ then

the size of the test is $\underline{\underline{\alpha}}$

$P_{\theta_0}(\Lambda(X) < K_0) \leq 1-\alpha < P_{\theta_0}(\Lambda(X) \leq K_0)$

$$\gamma = \frac{P_{\theta_0}(\Lambda(X) \leq K_0) - (1-\alpha)}{P_{\theta_0}(\Lambda(X) \leq K_0) - P_{\theta_0}(\Lambda(X) < K_0)}$$

Size $= \underline{\underline{\alpha}}$

Let ϕ_0 be a likelihood ratio test with $K, \gamma(x)$ and size α.
Suppose ϕ is a test of the same size α and the same power.

$$v(x) = (\phi_0(x) - \phi(x))(f(x;\theta_1) - Kf(x;\theta_0)) \qquad v(x) \geq 0$$

$$\int_{\mathbb{R}} v(x)\,dx \geq 0 \xRightarrow[\text{same power}]{\text{same size}} \int_{\mathbb{R}} v(x)\,dx = 0 \Rightarrow \begin{array}{c} \phi_0(x) = \phi(x) \\ \text{or} \\ \Lambda(x) = K \end{array} \quad \text{almost surely}$$

Example : $X \sim \text{Bin}(5, \theta)$, $\quad H_0 : \theta = 0.5$ **vs** $H_1 : \theta = 0.75$

a coin

fair $\theta_0 = 0.5$

$\theta_1 = 0.75$

α is given!

$\phi_0 :=$

We are going to determine the constant K and the function $\gamma(x)$

$$f(x;\theta) = \binom{5}{x} \theta^x (1-\theta)^{5-x}$$

$$\Lambda(x) = \frac{\theta_1^x (1-\theta_1)^{5-x}}{\theta_0^x (1-\theta_0)^{5-x}} = \left(\frac{\theta_1}{\theta_0}\right)^x \left(\frac{1-\theta_1}{1-\theta_0}\right)^{5-x} = \left(\frac{\frac{3}{4}}{\frac{1}{2}}\right)^x \left(\frac{1-\frac{3}{4}}{1-\frac{1}{2}}\right)^{5-x} = \left(\frac{3}{2}\right)^x \left(\frac{1}{2}\right)^{5-x} = 3^x 2^{-x} 2^{x-5} = 2^{-5} 3^x$$

$$P_{\theta_0}\{\Lambda(x) \le K\} = P_{\theta_0}\left\{X \le \frac{5\log 2 + \log K}{\log 3}\right\} = F_X\left(\frac{5\log 2 + \log K}{\log 3}; \theta_0\right)$$

$$\Lambda(x) \le K \Leftrightarrow 2^{-5} 3^x \le K \Rightarrow 3^x \le 2^5 K \Rightarrow x \log 3 \le 5\log 2 + \log K \Rightarrow \boxed{x \le \frac{5\log 2 + \log K}{\log 3}}$$

$x \in \{0,1,2,3,4,5\}$ $\quad \theta_0 = 0.5$

$\alpha = 0.05$ $1-\alpha = 0.95$

Example : $X \sim \text{Bin}(5, \theta)$, $H_0 : \theta = 0.5$ **vs** $H_1 : \theta = 0.75$

x	0	1	2	3	4	5
F	0.031	0.19	0.5	0.81	0.97	1

$1-\alpha = 0.95$

$$\frac{5 \log 2 + \log k_0}{\log 3} = 4$$

$$\log k_0 = 4 \log 3 - 5 \log 2 = \log \frac{3^4}{2^5}$$

$$\boxed{k_0 = \frac{3^4}{2^5} = 2.5125}$$

$$\gamma = \frac{0.97 - 0.95}{0.97 - 0.81} = 0.125$$

$$\phi_0(x) = \begin{cases} 1, & \Lambda(x) > 2.5125 \\ 0.125 & \Lambda(X) = 2.5125 \\ 0, & \underline{\text{otherwise}} \end{cases}$$