MEM-264 Applied Statistics

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Size of a test and its critical region

$\Theta = \Theta_0 \cup \Theta_1$, $\Theta_0 \cap \Theta_1 = \emptyset$

$H_0 \leftrightarrow \Theta_0$, $H_1 \leftrightarrow \Theta_1$

T: test statistics

data $\underset{\sim}{X} \longrightarrow t_{obs} \longrightarrow$ p-value

Test of size $\alpha \in (0, 1)$

if H_0 is a simple, $P\{\text{Reject } H_0\} = \alpha$

if $H_0 : \theta \in \Theta_0$ is valid

$$P\{\text{Reject} H_0\} \leq \alpha \text{ for all } \theta \in \Theta_0$$

In general $\exists!\ \theta_0 \in \Theta_0$ such that $P_{\theta_0}\{\text{Reject } H_0\} = \alpha$

α - Significance level (επίπεδο σημαντικότητας)

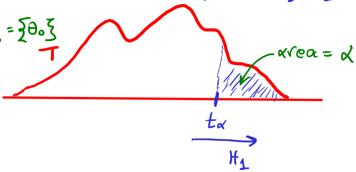
- In this course we will use almost everywhere $a = 0.05$

(κρίσιμη περιοχή)

Critical region $\mathcal{C}_\alpha$

in case of a simple H_0, H_1 one-sided

$\Theta_0 = \{\theta_0\}$



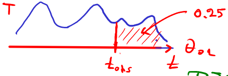
$$P\{t(\mathbf{X}) \in \mathcal{C}_\alpha\} \leq \alpha \text{ for all } \theta \in \Theta_0$$

Another example

$\Theta = \{\theta_{01}, \theta_{02}, \theta_1\}$

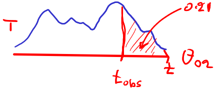
$\theta_{01} < \theta_{02} < \theta_1$

$\Theta_0 = \{\theta_{01}, \theta_{02}\}$, $\Theta_1 = \{\theta_1\}$



$\max\{0.25, 0.21\}$

$P\{\text{Reject } H_0\} \leq 0.25$

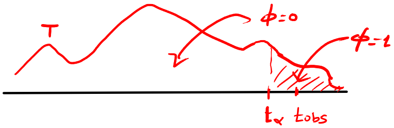


A test of size 0.25

Definition : Nonrandomized test function ϕ

A nonrandomized test function is a statistics ϕ from the sample space $\mathcal{X}$ to the set $\{0, 1\}$.

$$\phi(\mathbf{x}) = \begin{cases} 1, & t(\mathbf{x}) \in C_\alpha \quad \text{(reject } H_0\text{)} \\ 0, & \text{otherwise} \quad \text{(do not reject } H_0\text{)} \end{cases}$$



$$\phi : \mathcal{X} \to \{\overset{H_0}{0}, \overset{\text{Reject } H_0}{1}\}$$

$T(X) = X$ $\qquad$ (pmf)

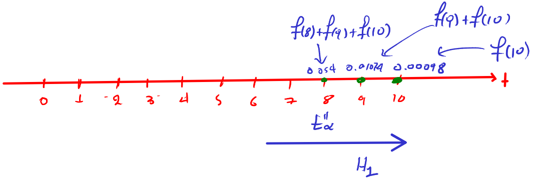
$f_T(t;\theta) = \binom{10}{t}\theta^t(1-\theta)^{10-t}$ $\qquad X \sim \text{Bin}(10, \theta)$ $\qquad f(x;\theta) = \binom{10}{x}\theta^x(1-\theta)^{10-x}$

$$H_0 : \theta = 0.5 \quad \text{vs} \quad H_1 : \theta \in (0.5, 1), \quad \alpha = 0.054$$

Let $\theta = 0.5$

$f(t) = \binom{10}{t}\left(\frac{1}{2}\right)^t\left(\frac{1}{2}\right)^{10-t} = \binom{10}{t} \cdot 2^{-10}$

$t_\alpha \in \{0, 1, \ldots, 10\}$



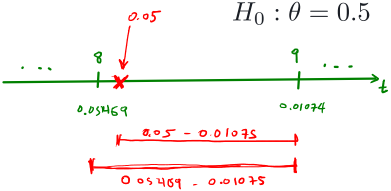
$\sum_{t \geq t_\alpha} f(t) = \alpha = 0.054$

$C_\alpha = C_{0.054} = \{8, 9, 10\}$

$\phi(x) = \begin{cases} 1, & x \in \{8,9,10\} \quad \text{Reject } H_0 \\ 0, & \text{otherwise} \quad \underline{H_0} \end{cases}$

$$X \sim \text{Bin}(10, \theta)$$

$$H_0 : \theta = 0.5 \quad \text{vs} \quad H_1 : \theta \in (0.5, 1), \quad \alpha = 0.05$$



$X \in \{9, 10\}$ Reject H_0

$X = 8$ Reject H_0 with probability $\dfrac{0.05 - 0.01075}{0.05469 - 0.01075} \approx 0.8933$

$X \in \{0, \dots, 7\}$ Do not reject H_0

Probability of rejection

$$\phi(x) = \begin{cases} 1, & x \in \{9, 10\} \\ 0.8933, & x = 8 \\ 0, & \text{otherwise} \end{cases}$$

According to the next frame

$$\phi : \mathcal{X} \to \{0, \gamma, 1\}, \quad \gamma \in (0,1)$$

Definition : Randomized test function ϕ

A randomize test function is a statistics ϕ from the sample space $\mathcal{X}$ to the set $\{0, \gamma, 1\}$ for some $\gamma \in (0, 1)$.

$$\phi(\mathbf{x}) = \begin{cases} 1, & t(\mathbf{x}) \in C_\alpha^{(1)} \quad \text{(reject } H_0\text{)} \\ \gamma, & t(\mathbf{x}) \in C_\alpha^{(\gamma)} \quad \text{(reject } H_0 \text{ with probability } \gamma\text{)} \\ 0, & \text{otherwise} \quad \text{(do not reject } H_0\text{)} \end{cases}$$

$X \sim Bin(10, \theta)$, $\theta \in (0,1)$

$H_0 : \theta \in (0, 0.5]$ vs $H_1 : \theta \in (0.5, 1)$

We have already solved for $H_0 : \theta = 0.5$

for $H_0 : \theta = 0.5$ and $\phi(x) = \begin{cases} 1, & x \in \{9,10\} \\ 0.8933, & x = 8 \\ 0, & \text{otherwise} \end{cases} \implies \alpha = 0.05$

$\theta^* < 0.5$ $\qquad f(x, \theta^*) < f(x, \theta = 0.5)$, $\quad x \geq 8$

$\cancel{\binom{10}{x}} (\theta^*)^x (1 - \theta^*)^{10-x} < \cancel{\binom{10}{x}} 2^{-10} \implies \alpha^* < \alpha$

for $H_0 : \theta \in (0, 0.5]$ $\qquad \phi(x) = \begin{cases} 1, & x \in \{9,10\} \\ 0.8933, & x = 8 \\ 0, & \text{otherwise} \end{cases}$.