MEM-264 Applied Statistics

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X r.v. $F_X(x)$ | X_1, X_2 $F_{X_1X_2}(x_1,x_2) = \mathbb{P}(\{X_1 \le x_1\} \cap \{X_2 \le x_2\}) = \mathbb{P}(X_1 < x_1, X_2 < x_2)$

Definition : Joint distribution function

Let $X_1, \dots, X_n$ be random variables. Their joint distribution function is defined as

$$F_{X_1 \dots X_n}(x_1, \dots, x_n) = P(X_1 \le x_1, \dots, X_n \le x_n)$$

Definition : Joint probability mass function

$\{X_1 = x_1\} \cap \dots \cap \{X_n = x_n\}$

$$f_{X_1, \dots, X_n}(x_1, \dots, x_n) = P(X_1 = x_1, \dots, X_n = x_n)$$

Definition : Joint probability density function $f_X(x) = F'_X(x)$

$$f_{X_1, \dots, X_n}(x_1, \dots, x_n) = \frac{\partial^n F_{X_1, \dots, X_n}(x_1, \dots, x_n)}{\partial x_1, \dots, \partial x_n}$$

X, Y r.v $\quad f_{XY}(x,y)$ known.

if Y is a discrete r.v then $f_X(x) = \sum_{y \in Y} f_{XY}(x,y)$

Definition : marginal mass function

$$\underbrace{\sum_{x_1} \sum_{x_2} \cdots \sum_{x_{i-1}} \sum_{x_{i+1}} \cdots \sum_{x_n}}_{n-1}$$

$$f_{X_i}(x_i) = \sum_{x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n} f_{X_1,\ldots,X_n}(x_1,\ldots,x_n)$$

Definition : marginal density function

$$f_{X_i}(x_i) = \underbrace{\int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_{i-1}} \int_{-\infty}^{x_{i+1}} \cdots \int_{-\infty}^{x_n}}_{n-1} f_{X_1,\ldots,X_n}(x_1,\ldots,x_n) dx_1 \cdots dx_{i-1} dx_{i+1} \cdots dx_n$$

$f_{X_1 X_2 \dots X_n}(x_1, \dots, x_n)$ $\quad f_{X_j}(x_j)$

$$f(x_1, \dots, x_n) = \prod_{j=1}^{n} f(x_j)$$

Definition

Two random variables X, Y are called **independent**, iff for every $x \in \mathcal{X}$ and $y \in \mathcal{Y}$,

$$f_{XY}(x, y) = f_X(x) f_Y(y)$$

Definition : Independent (ανεξάρτητες) and identically distributed (ισόνομες) random variables

We call that $X_1, \dots, X_n$ are i.i.d iff they are independent and obey the same distribution.

Normal distribution

variable

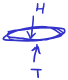
example : X, Y iid $\quad X \sim \mathcal{N}(0,1) \longrightarrow Y \sim \mathcal{N}(0,1)$ $\quad f_{XY}(x,y) = f_X(x) f_X(y) = f_Y(x) f_Y(y)$

mean value

- The value of the random variable that is expected to occur on average, if the number of repetitions of the random experiment tends to infinity.

Definition

Expected value (or mean) of a discrete random variable X is defined to be



a fair coin

Taie Head

$f_X(x) = \frac{1}{2}, \; x \in \{0, 1\}$

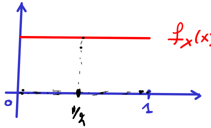
$$\mathbb{E}(X) = \mu_X = \sum_{x \in \mathcal{X}} x f_X(x)$$

$\mathbb{E}(X) = 0 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2} = \frac{1}{2}$

Definition

Expected value (or mean) of a continuous random variable X is defined to be

X is uniformly distributed on $[0,1]$



$f_X(x) = \mathbb{1}\{x \in [0,1]\}$

$$\mathbb{E}(X) = \mu_X = \int_{\mathcal{X}} x f_X(x) dx$$

$\mathbb{E}(X) = \int_{\mathbb{R}} x \, \mathbb{1}\{x \in [0,1]\} dx = \int_0^1 x \, dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$

Properties of expectations

- Let $X_1, X_2, \dots, X_n$ be random variables and $c_1, \dots, c_n$ constants, then

$$\mathbb{E}\left(\sum_{j=1}^{n} c_j X_j\right) = \sum_{j=1}^{n} c_j \mathbb{E}(X_j).$$

- Let $X_1, X_2, \dots, X_n$ be **independent (ανεξάρτητες)** random variables, then

$$\mathbb{E}\Bigg(\underbrace{\prod_{j=1}^{n} X_j}_{g(x_1,\dots,x_n)}\Bigg) = \prod_{j=1}^{n} \mathbb{E}(X_j).$$

$f(x_1,\dots,x_n)$ joint probability density or mass function.

$$\mathbb{E}\{g(x_1,\dots,x_n)\} = \int_{\mathcal{X}} \underbrace{g(x_1,\dots,x_n)}_{= x_1\cdots x_n} \overbrace{f(x_1,\dots,x_n)}^{f(x_1)f(x_2)\cdots f(x_n)} dx_1\dots dx_n$$

$$= \prod_j \int_{x_j} x_j f(x_j)\, dx_j = \prod_j \mathbb{E}(x_j)$$

~~**Definition**~~ Proposition.

Let $Y = g(X)$, where X is a random variable and g is a given function. Then Y is also a random variable and

$$\mathbb{E}(Y) = \mathbb{E}(g(X)) = \int_{\mathcal{X}} g(x) f_X(x) dx$$

$$\| \qquad \int_{\underline{Y}} y \, f_{\underline{Y}}(y) \, dy$$

example: $X \sim U[0,1]$ ↑ uniform $\quad Y = X^2 \quad \mathbb{E}(Y) = \int_0^1 x^2 dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}.$

Variance (διασπορά) of random variables

Definition

The **variance** of a random variable X with $|\mathbb{E}(X)| < \infty$ is defined as

$$\sigma^2 = \sigma_X^2 = \mathrm{Var}(X) = \mathbb{E}\{(X - \mathbb{E}(X))^2\}$$

.
The square root of the variance is called the **standard deviation (τυπική απόκλιση)** of the random variable and its denoted by σ or σ_X.

Properties of variances

- $\mathrm{Var}(X) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$
- $\mathrm{Var}(\alpha X + \beta) = \alpha^2 \mathrm{Var}(X)$ for any α, β constants
- If $X_1, \ldots, X_n$ are **independent** then $\mathrm{Var}(\sum_{j=1}^n X_j) = \sum_{j=1}^n \mathrm{Var}(X_j)$

$$\underset{\sim}{\Sigma} = \begin{bmatrix} \mathrm{Var}(X) & \mathrm{Cov}(X,Y) \\ \mathrm{Cov}(Y,X) & \mathrm{Var}(Y) \end{bmatrix}$$

Covariance of random variables

Let X, Y random variables then

$\mathrm{Cov}(X,X) = \mathrm{Var}(X)$

$$\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\boxed{\mathbb{E}((X - \mathbb{E}(X))(Y - \mathbb{E}(Y)))}$$

The last term is called the **covariance** of X, Y and we denote it as $\mathrm{Cov}(X, Y)$

Let X, Y be independent random variables then

- $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$
- $\text{Cov}(X, Y) = 0$ $\longrightarrow$ $\text{Cov}(X,Y) = \mathbb{E}\Big((X - \cancel{\mathbb{E}(X)})(Y - \mathbb{E}(Y))\Big) = \mathbb{E}\big(X(Y - \mathbb{E}(Y))\big) = \underset{0}{\mathbb{E}(XY)} - \cancel{\mathbb{E}(X\mathbb{E}(Y))}$

$\mathbb{1}\{x \in [0,1]\} = \begin{cases} 1, & \text{if } x \in [0,1] \\ 0, & \text{otherwise} \end{cases}$

Is the reverse true?

uniformly distributed on $[0,1]$

Counterexample: $X \sim U[-1,1]$ $\quad Y = X^2$ $\quad \mathbb{E}\{Y\} = \frac{1}{2}\int_{-1}^{1} x^2 = \frac{1}{6}x^3\Big|_{-1}^{1} \neq 0.$

$f_X(x) = \frac{1}{2}\, \mathbb{1}\{x \in [0,1]\}$

$\mathbb{E}\{X^3\} = \int_{-1}^{1} x^3 f(x)\,dx = \frac{1}{2}\int_{-1}^{1} x^3\,dx = \frac{1}{8}x^4\Big|_{-1}^{1} = \frac{1}{8} - \frac{1}{8} = 0$

$\mathbb{E}\{X\} = 0$

A binomial experiment must satisfy the conditions:

- There are N identical and independent repetitions of a random event
- In each repetition there are two possible outcomes (success or failure) of predetermined probabilities

Definition

Let $X \sim \text{Bin}(N, p)$ denote that the random variable X obeys the binomial distribution with parameters N and $p \in [0, 1]$. The probability mass function for $x \in \{0, \dots, N\}$ successes is given by

$P(X=x)$ "

$$f_X(x) = \binom{N}{x} p^x (1-p)^{N-x}$$

$$\binom{N}{x} = \frac{N!}{x!(N-x)!}$$

Example

Consider the experiment of 3 tosses of a fair coin. Let X be the number of heads.

$X \in \{0, 1, 2, 3\}$ $f_x(3)$

$f_x(0)$ $f_x(1)$ $f_x(2)$

$p = \frac{1}{2}$

$N = 3$

$f_x(x) = \binom{N}{x}\left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{3-x} = \frac{1}{8} \frac{3!}{x!(3-x)!}$