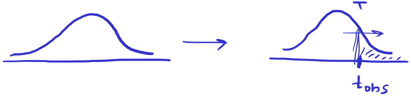
P-value

H_0 vs H_1

$\downarrow$

$H_0 : \theta = \theta_0$ $\qquad$ $H_1 : \theta > \theta_1$



Data $\rightarrow t_{obs}$

Power : $\mathbb{P}\{\text{Reject } H_0\} = \mathbb{E}_\theta\{\phi(X)\} = W(\theta)$

$\phi \leftarrow$ test function

Size α

$\mathbb{P}\{\text{Reject } H_0\} \le \alpha$ for all $\theta \in \Theta_0$

$\exists\, \theta' \in \Theta_0$ such that $\mathbb{P}_{\theta'}\{\text{Reject } H_0\} = \alpha$.

UMP of size α

$\mathbb{E}_\theta \phi_0(X) \le \alpha \quad \forall \theta \in \Theta_0$.

Given any other ϕ $\quad \mathbb{E}_\theta \phi(X) \le \alpha \quad \forall \theta \in \Theta_0$ and $\mathbb{E}_\theta \phi_0(X) \ge \mathbb{E}_\theta \phi(X) \quad \forall \theta \in \Theta_1$.

X_1, X_2, X_3 iid $X_1 \sim Bin(3, \theta)$ $H_0: \theta = 0.5$ vs $H_1: \theta \in (0.5, 1)$

0.5^2

0.5^3

$f(x_1, x_2, x_3; \theta) = f(\underset{\sim}{x}; \theta)$ where $\underset{\sim}{x} = (x_1, x_2, x_3)^T$

$f(\underset{\sim}{x}; \theta) = \prod_{j=1}^{3} f(x_j; \theta)$, $f(x_j; \theta) = \binom{3}{x_j} \theta^{x_j} (1-\theta)^{3-x_j}$

$= \theta^{\sum x_j} (1-\theta)^{9-\sum x_j} \prod_{j=1}^{3} \binom{3}{x_j} = L(\theta; \underset{\sim}{x})$

$\theta_0 < \theta_1$ $\Lambda(\underset{\sim}{x}) = \dfrac{f(\underset{\sim}{x}; \theta_1)}{f(\underset{\sim}{x}; \theta_0)} = \dfrac{\theta_1^{\sum x_j} (1-\theta_1)^{9-\sum x_j}}{\theta_0^{\sum x_j} (1-\theta_0)^{9-\sum x_j}} = \left(\dfrac{\theta_1}{\theta_0}\right)^{\sum x_j} \left(\dfrac{1-\theta_1}{1-\theta_0}\right)^{9-\sum x_j}$ $\theta_0 = 0.5$

$\Lambda(\underset{\sim}{x}) = (2\theta_1)^{\sum x_j} (2 - 2\theta_1)^{9-\sum x_j}$ $t(\underset{\sim}{x}) = \sum x_j$

$\tilde{\Lambda}(t(\underset{\sim}{x})) = (\underbrace{2\theta_1}_{>1})^{t(\underset{\sim}{x})} (\underbrace{2 - 2\theta_1}_{<1})^{9-t(\underset{\sim}{x})}$ $\theta_1 > 0.5 \Rightarrow 2\theta_1 > 1 \Rightarrow -2\theta_1 < -1$

$0 < 2 - 2\theta_1 < 2 - 1 = 1$

if $t_1 < t_2$ then $\tilde{\Lambda}(t_1) < \tilde{\Lambda}(t_2)$.

$(1,2,1) \rightarrow t_{obs} = 4$

$(1,2,1)$

$\phi(\underset{\sim}{X}) = \begin{cases} 1, & t(X) > t_0 \\ \gamma, & t(X) = t_0 \\ 0, & t(X) < t_0 \end{cases}$ $T = t(X) = \sum X_j \sim Bin(9, \theta)$

UMP test !!

$X_1, \dots, X_n$ iid $X_1 \sim Po(\theta)$

$\alpha = 0.05$

$H_0: \theta = 1$ vs $H_1: \theta = 2$

$$f(x_j;\theta) = \frac{\theta^{x_j} e^{-\theta}}{x_j!} \Rightarrow f(\underset{\sim}{x};\theta) = e^{-n\theta}\, \theta^{\sum x_j} \prod_{j=1}^{n} \frac{1}{x_j!}$$

$$\Lambda(\underset{\sim}{x}) = \frac{f(\underset{\sim}{x};2)}{f(\underset{\sim}{x};1)} = e^{-n}\, 2^{\sum x_j}$$

$t = \sum x_j \sim Po(n\theta)$

$\Lambda \uparrow$ with respect to $t(x)$

$$\phi_0(\underset{\sim}{x}) = \begin{cases} 1, & \text{if } \Lambda(\underset{\sim}{x}) > K \\ \gamma, & \text{if } \Lambda(\underset{\sim}{x}) = K \\ 0, & \text{if } \Lambda(\underset{\sim}{x}) < K \end{cases} \longrightarrow \phi_0(\underset{\sim}{x}) = \begin{cases} 1, & \text{if } t(\underset{\sim}{x}) > t_0 \\ \gamma, & \text{if } t(\underset{\sim}{x}) = t_0 \\ 0, & \text{if } t(\underset{\sim}{x}) < t_0 \end{cases}$$

ϕ_0 UMP of size α.

X

sample space $\mathcal{X} = \{2, \dots, 8\}$ $H_0 : \theta = \theta_0$ vs $H_1 : \theta = \theta_1$

x	2	3	4	5	6	7	8
$f(x;\theta_0)$	0.05	0.02	0.33	0.1	0.2	0.1	0.2
$f(x;\theta_1)$	0.01	0.3	0.01	0.18	0.2	0.2	0.1
$\Lambda(x)$	1/5	15	1/33	1.8	1	2	0.5

$$\phi_0(X) = \begin{cases} 1, & \Lambda(X) > k \\ \gamma, & \Lambda(X) = K \\ 0, & \Lambda(X) < K \end{cases}$$

$$\Lambda(4) < \Lambda(2) < \Lambda(8) < \Lambda(6) < \Lambda(5) < \Lambda(7) < \Lambda(3)$$

$$\begin{array}{ccccccc} \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array}$$

$$t(x) = \begin{cases} 2 & , x = 2 \\ 7 & , x = 3 \\ 1 & , x = 4 \\ 5 & , x = 5 \\ 4 & , x = 6 \\ 6 & , x = 7 \\ 3 & , x = 8 \end{cases}$$

$$t^{-1}(y) = \begin{cases} 4 & , y = 1 \\ 2 & , \underline{y = 2} \\ 8 & , y = 3 \\ 6 & , y = 4 \\ 5 & , y = 5 \\ 7 & , y = 6 \\ 3 & , y = 7 \end{cases}$$

$t^{-1}(t(4)) = 4$ $\quad Y \in \{1, \dots, 7\}$

$$t^{-1} \circ t(x) = x$$

$$\Lambda(x) = \Lambda(t^{-1} \circ t(x)) = \tilde{\Lambda}(t(x))$$

$\Lambda \nearrow$ with respect to t.

0.15

t	1	2	3	4	5	6	7
$f(t;\theta_0)$	0.33	0.05	0.2	0.2	0.1	0.1	0.02
$F(t,\theta_0)$	0.33	0.38	0.58	0.78	0.88	0.98	1
$\tilde{\Lambda}(t)$	1/33	1/5	0.5	1	1.8	2	15

$$\phi_0(x) = \begin{cases} 1, & t(x) > 6 \\ \gamma, & t(x) = 6 \\ 0, & t(x) < 6 \end{cases} = \begin{cases} 1, & t(x) > 6 \\ 0.3, & t(x) = 6 \\ 0, & t(x) < 6 \end{cases}$$

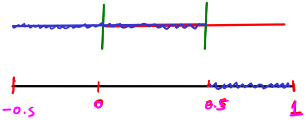
$$\gamma = \frac{0.98 - 0.95}{0.98 - 0.88} = \frac{0.03}{0.1} = 0.3$$

$$\phi_0(x) = \begin{cases} 1, & x = 3 \\ 0.3, & x = 7 \\ 0, & x \in \{2, 4, 5, 6, 8\} \end{cases}$$

X continuous r.v $x \in [-0.5, 1]$

$H_0: \underline{\theta = 0}$ $\quad X \sim U[-0.5, 0.5]$ $\quad f(x; \theta=0) = 1\{x \in [-0.5, 0.5]\}$ —

$H_1: \theta = 1$ $\quad x \sim U[0, 1]$ $\quad f(x; \theta=1) = 1\{x \in [0, 1]\}$ —



$$\Lambda(x) = \frac{f(x; \theta=1)}{f(x; \theta=0)} = \begin{cases} \infty, & x \in (0.5, 1] \\ 1, & x \in [0, 0.5] \\ 0, & x \in [-0.5, 0) \end{cases}$$

$\underline{\alpha = 0.05}$

$$T = t(X) = X$$

$$\phi_0(x) = \begin{cases} 1, & t > t_1 \\ \gamma, & t \in (t_0, t_1] \\ 0, & t \leq t_0 \end{cases}$$

$t_1 = 0.5$

$t_0 = 0$

$$E_{\theta_0}\{\phi_0(X)\} = P_{\theta_0}\{t > t_1\} + \gamma \cdot P_{\theta_0}\{t \in (t_0, t_1]\} = 0.05$$

$$= 0 + \gamma \cdot 0.5 = 0.05 \Rightarrow \gamma = \frac{0.05}{0.5} = 0.1$$

$$\phi_0(X) = \begin{cases} 1 , & t > 0.5 \\ 0.1 , & t \in [0, 0.5] \\ 0 , & t \leq 0 \end{cases}$$