MEM-264 Applied Statistics

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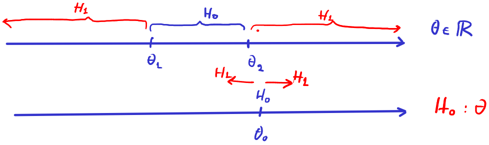


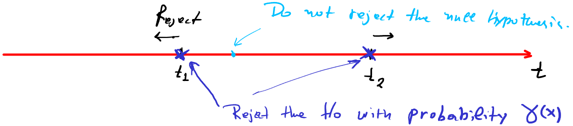
special case: $\theta_1 \to \theta_2 = \theta_0$

$H_0 : \theta = \theta_0$ vs $H_1 : \theta \neq \theta_0$.

- $H_0 : \theta \in \Theta_0 = [\theta_1, \theta_2]$ vs $H_1 : \theta \in \Theta_1 = \Theta - \Theta_0$
- If we have a distribution that is MLR with respect to $t(X)$, we can expect tests of the following form:

$$\phi(x) = \begin{cases} 1, & \text{if } \; t(x) < t_1 \text{ or } t(x) > t_2 \\ \gamma(x), & \text{if } \; t(x) \in \{t_1, t_2\} \\ 0, & \text{if } \; t_1 < t(x) < t_2 \end{cases}$$

← two-sided test function with respect to a statistic $t(X)$



When we study two-sided hypotheses, We need the concept of the unbiasedness.

Definition : Unbiased test functions

A test function ϕ is called unbiased of size of α if

$$\sup_{\theta \in \Theta_0} \mathbb{E}_\theta\{\phi(X)\} \leq \alpha$$

and

$$\mathbb{E}_\theta\{\phi(X)\} \geq \alpha \text{ for all } \theta \in \Theta_1$$

- An unbiased test ensures that the probability of rejecting the null hypothesis is higher when H_0 is false that when it is true !!
- **A test which is UMP amongst the class of all unbiased tests is called uniformly most powerful unbiased (UMPU) test.**
- Unbiasedness is not an optimality criterion.
- It is simply used to reduce the complexity of the problem.

* Normal
* Poisson
* Binomial (with N fixed)

belong to the exponential family!

We said that a random variable X belongs to an one-dimensional exponential family iff its probability density (or mass) function is of the following form:

$$f(x;\theta) = c(\theta)h(x)\exp\{\theta t(x)\}$$

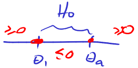
The $T = t(X)$ is called the natural statistics of X.

- We will restrict our analysis to continuous random variables X to avoid the need for randomized test functions.

Theorem (without proof) $\Theta_0 = [\theta_1, \theta_2] \subseteq \mathbb{R}$ $\Theta_1 = (-\infty, \theta_1) \cup (\theta_2, \infty)$

Let ϕ be any test function. There exist a unique two-sided test function ϕ_0 which is a function of $T = t(X)$ such that

$$\mathbb{E}_{\theta_j}\phi_0(X) = \mathbb{E}_{\theta_j}\phi(X),\ j = 1, 2$$



and

$$\mathbb{E}_\theta\phi_0(X) - \mathbb{E}_\theta\phi(X) = \begin{cases} \leq 0, & \theta \in (\theta_1, \theta_2) \\ \geq 0, & \theta < \theta_1 \text{ or} \theta > \theta_2 \end{cases}$$

Colollary

Given $\alpha > 0$ $\alpha \in (0, 1]$, there exists a UMPU test of size α, that is of two-sided form in $t(X)$.

$(\mathbb{E}_\theta \phi(X) = \alpha)$

$\phi(X) = \alpha$. For any x we reject the null hypothesis with probability α

$$\mathbb{E}_\theta \phi_0(X) = \begin{cases} \alpha, & \text{if } \theta \in \{\theta_1, \theta_2\} \\ \leq \alpha, & \text{if } \theta \in (\theta_1, \theta_2) \\ \geq \alpha, & \text{if } \theta < \theta_1 \text{ or } \theta > \theta_2. \end{cases}$$

Let ϕ be any unbiased test function of size α.

$\sup_{\theta \in \Theta_0} \mathbb{E}_\theta \phi(X) \leq \alpha$ continuity of $\mathbb{E}\phi(X)$ $\Rightarrow$ $\mathbb{E}_{\theta_1}\phi(X) = \mathbb{E}_{\theta_2}\phi(X) = \alpha$.

$\mathbb{E}_\theta \phi(X) \geq \alpha \quad \forall \theta \in \Theta_1$

Apply the previous theorem for $\phi(x)$ and $\phi_0(x)$.

Therefore ϕ_0 is UMPU

$\theta_1 \to \theta_2 = \theta_0$

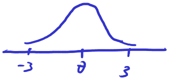
$$\mathbb{E}_\theta \phi_0(X) = \begin{cases} \alpha, & \text{if } \theta = \theta_0 \\ \geq \alpha, & \text{if } \theta \neq \theta_0 \end{cases}$$

$$H_0 : \theta = \theta_0 \quad \text{vs} \quad H_1 : \theta \neq \theta_0$$

There exists a two-sided test function for which:

$$\mathbb{E}_{\theta_0} \phi_0(X) = \alpha, \quad \text{and} \quad \frac{d}{d\theta} \mathbb{E}_\theta \phi_0(X)|_{\theta=\theta_0} = 0$$

- **Such a test is UMPU**
- The derivative existence follows from the assumption of exponential family !!



Consider the test function

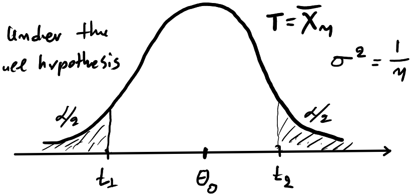
$$\phi(\underset{\sim}{x}) = \begin{cases} 1, & t(\underset{\sim}{x}) < t_1 \text{ or } t(\underset{\sim}{x}) > t_2 \\ 0, & t_1 \le t(\underset{\sim}{x}) \le t_2 \end{cases}$$

$$f(\underset{\sim}{x};\theta) \longrightarrow T = t(x) = \bar{X}_n \leftarrow (\text{natural statistics})$$

$$T \sim \mathcal{N}\left(\theta, \frac{1}{n}\right)$$

$$H_0 : \theta = \theta_0 \quad \text{vs} \quad H_1 : \theta \neq \theta_0$$

Under the null hypothesis



$$z_1 = \frac{t_1 - \theta_0}{\frac{1}{\sqrt{n}}} = \sqrt{n}(t_1 - \theta_0)$$

$$Z = \sqrt{n}(T - \theta_0) \sim \mathcal{N}(0,1)$$

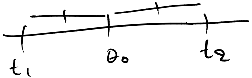
$$z_2 = \sqrt{n}(t_2 - \theta_0)$$

$$t_1 = \theta_0 + z_1/\sqrt{n} \qquad \mathbb{P}(Z \le z_1) = \alpha/2$$

$$t_2 = \theta_0 + z_2/\sqrt{n} \qquad \mathbb{P}(Z \le z_2) = 1 - \alpha/2$$

$$E_{\theta_0}\phi(X) = \alpha \qquad \frac{d}{d\theta} E_\theta\, \phi(X)\Big|_{\theta=\theta_0} = 0$$

$$\phi(\underset{\sim}{X}) = \begin{cases} 1, & t(\underset{\sim}{X}) < t_1 \text{ or } t(\underset{\sim}{X}) > t_2 \\ 0, & \text{otherwise} \end{cases}$$



$$E_{\theta_0}\phi(\underset{\sim}{X}) = 1\Big(\overset{\alpha/2}{P\{t(\underset{\sim}{X}) < t_1; \theta_0\}} + \overset{\alpha/2}{P\{t(\underset{\sim}{X}) > t_2; \theta_0\}}\Big) = \alpha \quad \checkmark$$

$$E_\theta\, \phi(\underset{\sim}{X}) = P\{t(\underset{\sim}{X}) < t_1; \theta\} + P\{t(\underset{\sim}{X}) > t_2; \theta\}$$

$$= P\{Z \le \sqrt{n}(t_1 - \theta)\} + 1 - P\{Z \le \sqrt{n}(t_2 - \theta)\}$$

$$= \Phi(\sqrt{n}(t_1 - \theta)) + 1 - \Phi(\sqrt{n}(t_2 - \theta))$$

$$\frac{d}{d\theta} E_\theta\, \phi(\underset{\sim}{X})\Big|_{\theta_0} = \sqrt{n}\,\Phi'(\sqrt{n}(t_1 - \theta_0)) - \sqrt{n}\,\Phi'(\sqrt{n}(t_2 - \theta_0)) = 0$$

where $\Phi(t) = P(T \le t)$

$$\Phi(t) = \int_0^t \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}s^2}\, ds \qquad \Phi'(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2}$$

As a result, $\phi \equiv \phi_0$

UMPU