MEM-264 Applied Statistics

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$H_0 : \theta \in \{0, 1\}$ 2
vs) $2^2 = 4$
$H_1 : \theta \in \{2, 3\}$ 2

1. $H_0 : \theta = 0$ vs $H_1 : \theta = 2$
2. $H_0 : \theta = 0$ vs $H_1 : \theta = 3$
3. $H_0 : \theta = 1$ vs $H_1 : \theta = 2$
4. $H_0 : \theta = 1$ vs $H_1 : \theta = 3$

The following hypothesis testing problem

- Composite H_0 vs Composite H_1.

can be decomposed to

- Simple H_0 vs Simple H_1 for every $\theta_0 \in \Theta_0$ and $\theta_1 \in \Theta_1$.

Definition : Uniformy Most Powerful test

A uniformly most powerful test of size α is a test ϕ_0 for which:

- $\mathbb{E}_\theta \phi_0(X) \leq \alpha$ for all $\theta \in \Theta_0$.
- Given any other test ϕ for which $\mathbb{E}_\theta \phi(X) \leq \alpha$ for all $\theta \in \Theta_0$, we have $\mathbb{E}_\theta \phi_0(X) \geq \mathbb{E}_\theta \phi(X)$ for all $\theta \in \Theta_1$.

In many cases UMP tests do not exist.

$f(\underset{\sim}{x};\theta)$

Definition

$\Lambda(x)$

The family of densities $f(x;\theta)$, $\theta \in \Theta \subseteq \mathbb{R}$ is called to be of monotone likelihood ratio if exists a function $t(x)$ such that the likelihood ratio is a non-decreasing function of $t(x)$ whenever $\theta_1 \leq \theta_2$.

(ανεξάρτητες και ισόνομες)

Example $X_1, \ldots, X_n$ **i.i.d sample with** $f(x_1;\theta) = c(\theta)h(x_1)\underline{e^{\theta\tau(x_1)}}$

$$f(x_1,\ldots,x_n;\theta) = \prod_{j=1}^{n} f(x_j;\theta) = c^n(\theta)\, e^{\theta\sum_{j=1}^{n}\tau(x_j)} \prod_{j=1}^{n} h(x_j)$$

for $\theta_1 \leq \theta_2$

$$\frac{f(\underset{\sim}{x};\theta_2)}{f(\underset{\sim}{x};\theta_1)} = \Bigg[\overbrace{\frac{c(\theta_2)}{c(\theta_1)}}^{\text{constant}}\Bigg]^n e^{\overbrace{(\theta_2-\theta_1)}^{\geq 0}\sum_{j=1}^{n}\tau(x_j)} = \Lambda(\underset{\sim}{x}) = \tilde{\Lambda}(t(\underset{\sim}{x}))$$

We can set $t(\underset{\sim}{x}) = \sum_{j=1}^{n}\tau(x_j)$

If $\underset{t_1}{t(\underset{\sim}{x_1})} < \underset{t_2}{t(\underset{\sim}{x_2})}$ then $\tilde{\Lambda}(t_1) \leq \tilde{\Lambda}(t_2)$. Therefore, $f(\underset{\sim}{x};\theta)$ is of monotone likelihood ratio (MLR)

$t_1 < t_2$

Theorem

Suppose X has a distribution from a family which is MLR with respect to a fuction $t(x)$. In addition, let's suppose that the $T = t(X)$ is a continuous random variable. For the test problem:

$$H_0 : \theta \leq \theta_0 \quad \text{vs} \quad H_1 : \theta > \theta_0$$

α)
- The test function

$$\phi_0(x) = \begin{cases} 1, & \text{if } t(x) > t_0 \\ 0, & \text{if } t(x) \leq t_0 \end{cases}$$

  is UMP among all tests of size $\leq \mathbb{E}_{\theta_0}\{\phi_0(X)\}$.

β)
- Given the size α, there exists some t_0 such that the test ϕ_0 has size exactly α.

α)

$\underline{\theta_1 > \theta_0}$ $H_0: \theta = \theta_0$ vs $H_1: \theta = \theta_1$

$(t \leftrightarrow \lambda)$ $\leftarrow$ Neyman-Pearson (optimality)

$(t_0 \leftrightarrow k)$

The Neyman-Pearson test is of form $\phi_0(x) = \begin{cases} 1, & \text{if } t(x) > t_0 \\ 0, & \text{if } t(x) \le t_0 \end{cases}$

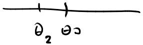
ϕ_0 is independent of θ_1, therefore $\underline{H_0: \theta = \theta_0 \text{ vs } H_1: \theta > \theta_0}$ ⊛

$\underline{\phi_0(x)} = \begin{cases} 1, & \text{if } t(x) > t_0 \\ 0, & \text{if } t(x) \le t_0 \end{cases}$ is the UMP

(Let ϕ be any other test and $\mathbb{E}_{\theta_0}\phi(x) \le \mathbb{E}_{\theta_0}\phi_0(x)$ then ϕ_0 has maximum power)

Next step

$H_0: \theta \le \theta_0$ vs $H_1: \theta > \theta_0$



Let $\theta_2 < \theta_0$. We are going to prove that $\boxed{\mathbb{E}_{\theta_2}\phi_0(x) \le \mathbb{E}_{\theta_0}\phi_0(x)}$

$\phi(x) = \mathbb{E}_{\theta_2}\phi_0(x)$, $\forall x$.

Independently of the value of x, we reject the null hypothesis with probability $\mathbb{E}_{\theta_2}\{\phi_0(x)\}$

$\mathbb{E}_{\theta_0}\{\phi(x)\} = \mathbb{E}_{\theta_2}\{\phi_0(x)\}$

From Neyman-Pearson $\Rightarrow$ $\mathbb{E}_{\theta_0}\{\phi_0(x)\} \geq \underline{\mathbb{E}_{\theta_0}\{\phi(x)\}} = \mathbb{E}_{\theta_2}\{\phi_0(x)\}$

$\mathbb{E}_{\theta_0}\{\phi_0(X)\} \geq \mathbb{E}_{\theta_2}\{\phi_0(X)\}$.

(*) if we construct the ϕ_0 to have size α

$$\mathbb{E}_{\theta_0}\{\phi_0(X)\} = \alpha \Rightarrow \mathbb{E}_{\theta}\{\phi_0(X)\} \leq \mathbb{E}_{\theta_0}\{\phi_0(X)\} = \alpha \quad \forall \theta \leq \theta_0$$

$H_0: \theta = \theta_0$ vs $H_1: \theta > \theta_0$ $\qquad \mathbb{E}_{\theta_0}\{\phi_0(X)\} = \alpha$

$\Downarrow$

$H_0: \theta \leq \theta_0$ vs $H_1: \theta > \theta_0$ $\qquad \mathbb{E}_{\theta}\{\phi_0(X)\} \leq \alpha, \ \forall \theta \leq \theta_0$

Therefore, ϕ_0 is UMP test among all test of size α.

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b)

Since $T = t(X)$ is a continuous random variable $\exists\, t_0$ st $P\{t(X) > t_0\} = \alpha$.