MEM-262 Παραμετρική Στατιστική

Τμήμα Μαθηματικών και Εφ. Μαθηματικών, Πανεπιστήμιο Κρήτη

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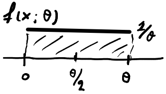
$\theta = 10 \quad x = 3 \xrightarrow{d_2} \hat{\theta} = 2 \cdot 3 = 6 \quad |10 - 6| = 4 \cdot 10^3 €$

Έστω τυχαία μεταβλητή $X \sim U[0, \theta]$, όπου $\theta \in \Theta = (0, \infty)$ και χώρο αποφάσεων $\mathcal{A} = \Theta$. Επιπλέον θεωρούμε την συνάρτηση απώλειας $L(\theta, d) = |\theta - d|$. Για την οικογένεια συναρτήσεων αποφάσεως $d_\alpha(X) = \alpha X, \ \alpha > 1$, βρείτε την τιμή του α ώστε η d_α να είναι παραδεκτή.

$d_\alpha(x) = \alpha X \quad \mathbb{E}_\theta\{d_2\} = \alpha\, \theta/2 \quad \alpha = 2$



$d_2(X) = 2X \quad \mathbb{E}_\theta\{d_2(X)\} = 2\, \mathbb{E}_\theta\{X\} = 2 \frac{\theta}{2} = \theta$

$$R(\theta, d_\alpha) = \mathbb{E}_\theta\left(L(\theta, d_\alpha(X))\right) = \int_{-\infty}^{+\infty} L(\theta, d_\alpha(X)) \cdot f(x;\theta)\, dx =$$

$$= \underbrace{\int_0^\theta \frac{1}{\theta} |\theta - \alpha x|\, dx}_{\parallel}$$

$\theta - \alpha x > 0 \Leftrightarrow \alpha x < \theta \Leftrightarrow x < \frac{\theta}{\alpha} < \theta$

$\theta - \alpha x < 0 \Leftrightarrow x > \frac{\theta}{\alpha}$

$$\int_0^{\theta/\alpha} \frac{1}{\theta}(\theta - \alpha x)\, dx - \int_{\theta/\alpha}^{\theta} \frac{1}{\theta}(\theta - \alpha x)\, dx =$$

$$= \int_0^{\theta/\alpha} dx - \int_{\theta/\alpha}^{\theta} dx + \frac{\alpha}{\theta}\int_{\theta/\alpha}^{\theta} x\, dx - \frac{\alpha}{\theta}\int_0^{\theta/\alpha} x\, dx =$$

$$= \theta/\alpha - \theta + \theta/\alpha + \frac{\alpha}{\theta}\left[\frac{x^2}{2}\right]_{\theta/\alpha}^{\theta} - \frac{\alpha}{\theta}\left[\frac{x^2}{2}\right]_0^{\theta/\alpha} =$$

$$= \frac{2}{\alpha}\theta - \theta + \frac{\alpha}{\cancel{\theta}}\left[\frac{\theta^{\cancel{2}}}{2} - \frac{\theta^{\cancel{2}}}{2\alpha^2}\right] - \frac{\cancel{\alpha}}{\cancel{\theta}}\frac{\theta^{\cancel{2}}}{2\alpha^{\cancel{2}}}$$

$$= \frac{2}{\alpha}\vartheta - \vartheta + \frac{\alpha\vartheta}{2} - \frac{\vartheta}{\alpha} = \vartheta\underbrace{\left(\frac{1}{\alpha} - 1 + \frac{\alpha}{2}\right)}_{q(\alpha)} = \vartheta q(\alpha)$$

$$R(\vartheta, d_\alpha) = \vartheta q(\alpha)$$

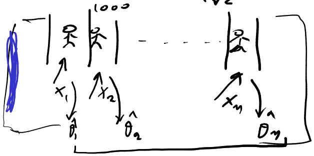
$\alpha > 1$

$$\partial_\alpha R(\vartheta, d_\alpha) = \vartheta q'(\alpha) = \vartheta\left(-\frac{1}{\alpha^2} + \frac{1}{2}\right) = 0 \Rightarrow \alpha^2 = 2 \Rightarrow \alpha = \sqrt{2} \approx 1.4$$

$$\boxed{d_{\sqrt{2}}(x) = \sqrt{2}\, x.}$$

$3 \longrightarrow 3\sqrt{2}$

$d_{\sqrt{2}}$

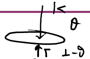


d_2

Άσκηση 1

Άσκηση 2

$\bar{X}_n = \frac{1}{n}\sum_{j=1}^{n} X_j \qquad d_\alpha(x) = \alpha(n)\,\bar{X}_n$



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Έστω $i.i.d$ τυχαίες μεταβλητές $X_1, X_2, \ldots, X_n \sim \mathrm{Be}(\theta)$, όπου $\theta \in \Theta = [0,1]$ και χώρο αποφάσεων $\mathcal{A} = \Theta$. Επιπλέον θεωρούμε την συνάρτηση απώλειας $L(\theta, d) = (\theta - d)^2$. Για την οικογένεια συναρτήσεων αποφάσεως $d_\alpha(X) = \alpha \bar{X}_n,\ \alpha > 0$, βρείτε την τιμή (ή τις τιμές) του α ώστε d_α να είναι Bayes όταν $\theta \sim U[0,1]$.

prior distribution

$\mathbb{E}(\bar{X}_n) = \frac{1}{n}\sum_{j=1}^{n}\mathbb{E}(X_j) = \frac{1}{n}\sum_{j=1}^{n}\mathbb{E}(X_1) = \mathbb{E}(X_1) = \theta$

$$R(\theta, d_\alpha) = \mathbb{E}_\theta\left((\theta - \alpha\bar{X}_n)^2\right) = \mathbb{E}_\theta\left(\theta^2 - 2\theta\alpha\bar{X}_n + \alpha^2(\bar{X}_n)^2\right) = \theta^2 - 2\theta\alpha\underbrace{\mathbb{E}(\bar{X}_n)}_{} + \alpha^2\,\mathbb{E}_\theta\left((\bar{X}_n)^2\right)$$

$$\mathbb{E}_\theta\left\{\frac{1}{n^2}\left(\sum_{j=1}^{n}X_j\right)^2\right\} = \frac{1}{n^2}\mathbb{E}_\theta\left\{\sum_{j=1}^{n}X_j^2 + 2\sum_{i\neq j}X_iX_j\right\} = \frac{1}{n^2}\left\{\overbrace{\sum_{j=1}^{n}\mathbb{E}_\theta(X_j^2)}^{n\mathbb{E}(X_1^2)} + \overbrace{2\sum_{j\neq i}\mathbb{E}(X_i)\mathbb{E}(X_j)}^{2\frac{n(n-1)}{2}(\mathbb{E}(X_1))^2}\right\} =$$

$\mathbb{E}(X_1) = \theta$

$$= \frac{1}{n^2}\left\{n\theta + n(n-1)\theta^2\right\}$$

$\mathbb{E}(X^2) = \sum x^2 P(x) = 1^2\cdot\theta + 0^2\cdot(1-\theta) = \theta$

$$R(\theta, d_\alpha) = \theta^2 - 2\alpha\theta^2 + \frac{\alpha^2}{\cancel{n^2}}\left\{\cancel{n}\theta + \cancel{n}(n-1)\theta^2\right\} =$$

$$= \theta^2\underbrace{\left(1 - 2\alpha + \alpha^2\frac{n-1}{n}\right)}_{g_n(\alpha)} + \underbrace{\frac{\alpha^2}{n}\theta}_{W_n(\alpha)}$$

Άσκηση 2

$$r(\pi,d_\alpha)=\int_0^1\left(\theta^2 q_n(\alpha)+\theta W_n(\alpha)\right)\cdot\overset{\pi(\theta)}{1}\cdot d\theta=\frac{1}{3}q_n(\alpha)+\frac{1}{2}W_n(\alpha)$$

$$\partial_\alpha r(\pi,d_\alpha)=\frac{1}{3}q_n'(\alpha)+\frac{1}{2}W_n'(\alpha)=0 \qquad \ldots$$

$$\Rightarrow \quad \alpha(n)=\frac{2n}{2n+1} \qquad \text{άρα} \qquad d_{\alpha(n)}(X)=\frac{2n}{2n+1}\bar{X}_n \qquad \alpha>0$$

$$E_\theta\left(d_{\alpha(n)}(X)\right)=\frac{2n}{2n+1}E(\bar{X}_n)=\frac{2n}{2n+1}E(X_1)=\frac{2n}{2n+1}\theta\neq\theta \quad \forall n.$$

Για $\alpha=1$ θα είναι αμερ. εκτιμήτρια.

$n=1$

$$\frac{2}{3}\cdot X_1 \quad \begin{cases} X_1=1 & \hat{\theta}=2/3 \\ X_1=0 & \hat{\theta}=0 \end{cases}$$

$\alpha=1$

$$E_\theta\left(d_\alpha(X)\right)=\alpha E(\bar{X}_n)=$$

$$=\alpha E(X_1)=\alpha\cdot\theta\overset{\alpha=1}{=}\theta$$

Άσκηση 2